▸▸▸ PROJECT FOR SECTION 8.2

Traffic Flow

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Network analysis, as we saw in the discussion of Kirchhoff's point and loop rules in Section 8.2, plays an important role in electrical engineering. In recent years, the concepts and tools of network analysis have been found to be useful in many other fields, such as information theory and the study of transportation systems. The following analysis of traffic flow through a road network during the peak period illustrates how systems of linear equations with many solutions can arise in practice.

Consider the typical road network of Figure 1. It represents an area of downtown Jacksonville, Florida. The streets are all one-way, with the arrows indicating the direction of traffic flow. The flow of traffic in and out of the network is measured in terms of vehicles per hour (vph). The figures given here are based on midweek peak traffic hours, 7 A.M. to 9 A.M. and 4 P.M. to 6 P.M. An increase of 2 percent in the overall flow should be allowed for during the Friday evening traffic flow. Let us construct a mathematical model that can be used to analyze this network.

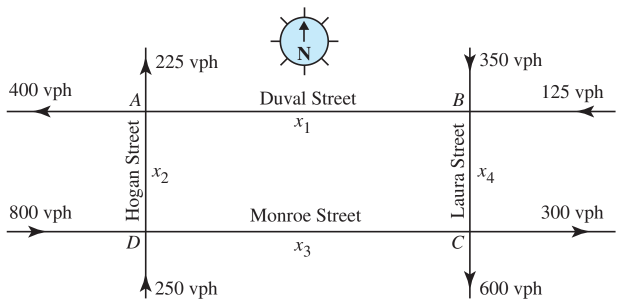

Figure 1 Downtown Jacksonville, Florida

Assume that the following traffic law applies:

All traffic entering an intersection must leave that intersection.

This conservation of flow constraint (compare it to Kirchhoff's point rule) leads to a system of linear equations:

Intersection A: Traffic in $= x_1 + x_2$.
Traffic out $= 400 + 225$. Thus $x_1 + x_2 = 625$.

Intersection B: Traffic in $= 350 + 125$.
Traffic out $= x_1 + x_4$. Thus $x_1 + x_4 = 475$.

Intersection C: Traffic in $= x_3 + x_4$.
Traffic out $= 600 + 300$. Thus $x_3 + x_4 = 900$.

Intersection D: Traffic in $= 800 + 250$.
Traffic out $= x_2 + x_3$. Thus $x_2 + x_3 = 1050$.

These constraints on the traffic are described by the following system of linear equations:

$$\begin{aligned} x_1 + x_2 &= 625 \\ x_1 + x_4 &= 475 \\ x_3 + x_4 &= 900 \\ x_2 + x_3 &= 1050 \end{aligned}$$

The method of Gauss-Jordan elimination can be used to solve this system of equations. The augmented matrix and reduced row-echelon form of the above system are as follows:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 625 \\ 1 & 0 & 0 & 1 & 475 \\ 0 & 0 & 1 & 1 & 900 \\ 0 & 1 & 1 & 0 & 1050 \end{pmatrix} \xRightarrow{\text{Row operations}} \begin{pmatrix} 1 & 0 & 0 & 1 & 475 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & 1 & 900 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The system of equations that corresponds to this reduced row-echelon form is

$$\begin{aligned} x_1 + x_4 &= 475 \\ x_2 - x_4 &= 150 \\ x_3 + x_4 &= 900. \end{aligned}$$

Expressing each leading variable in terms of the remaining variable, we get

$$\begin{aligned} x_1 &= -x_4 + 475 \\ x_2 &= x_4 + 150 \\ x_3 &= -x_4 + 900. \end{aligned}$$

As perhaps might be expected, the system of equations has many solutions—many traffic flows are possible. A driver does have a certain amount of choice at intersections. Let us now use this mathematical model to develop more information about the traffic flow. Suppose it becomes necessary to perform road work on the stretch DC of Monroe Street. It is desirable to have as small a traffic flow x_3 as possible along this stretch of road. The flows can be controlled along various branches by means of traffic lights. What is the minimum value of x_3 along DC that would not lead to traffic congestion? We use the preceding system of equations to answer this question.

All traffic flows must be nonnegative (a negative flow would be interpreted as traffic moving in the wrong direction on a one-way street). The third equation in the system tells us that x_3 will be a minimum when x_4 is as large as possible, as long as it does not exceed 900. The largest value x_4 can be without causing negative values of x_1 or x_2 is 475. Thus the smallest value of x_3 is $-475 + 900$, or 425. Any road work on Monroe Street should allow for traffic volume of at least 425 vph.

In practice, networks are much vaster than the one discussed here, leading to larger systems of linear equations that are handled on computers. Various values of variables can be entered into a computer to create different scenarios.

Related Problems

1. Construct a mathematical model that describes the traffic flow in the road network depicted in Figure 2. All streets are one-way streets in the directions indicated. The units are vehicles per hour. Give two distinct possible flows of traffic. What is the minimum possible flow that can be expected along branch AB?

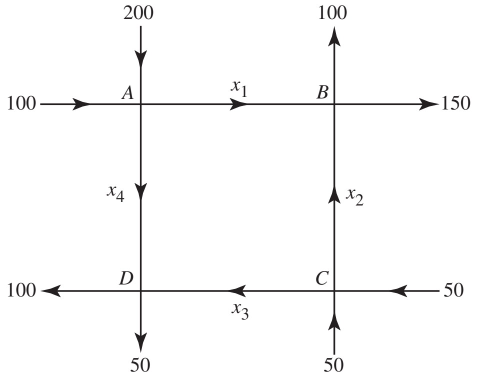

Figure 2 Traffic flow in Problem 1

2. Figure 3 represents the traffic entering and leaving a "roundabout" (rotary) road junction. Such junctions are very common in Europe. Construct a mathematical model that describes the flow of traffic along the various branches. What is the minimum flow theoretically possible along the branch BC? What are the other flows at that time? (The units of flow are vehicles per hour.)

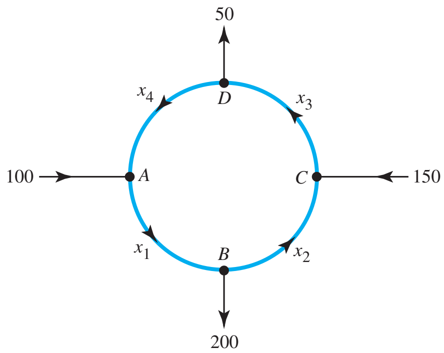

Figure 3 Traffic flow in Problem 2

3. Figure 4 represents the traffic entering and leaving another type of roundabout road junction in continental Europe. Such roundabouts ensure the continuous smooth flow of traffic at road junctions. Construct linear equations that describe the flow of traffic along the various branches. Use these equations to determine the minimum flow possible along x_1. What are the other flows at that time? (It is not necessary to compute the reduced row-echelon form. Use the fact that the traffic flow cannot be negative.)

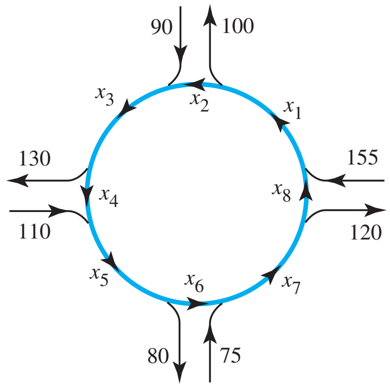

Figure 4 Traffic flow in Problem 3

4. Figure 5 describes a flow of traffic, with the units being vehicles per hour.

   (a) Construct a system of linear equations that describes this flow.

   (b) The total time it takes the vehicles to travel any stretch of road is proportional to the traffic along that stretch. For example, the total time it takes x_1 vehicles to traverse AB is kx_1 minutes. Assuming that the constant is the same for all sections of road, the total time for all 200 vehicles to be in this network is $kx_1 + 2kx_2 + kx_3 + 2kx_4 + kx_5$. What is this total time if $k = 4$? Give the average time for each car.

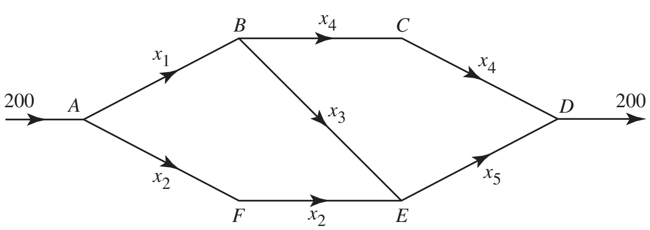

Figure 5 Traffic flow in Problem 4