PROJECT FOR SECTION 8.1

Two-Ports in Electrical Circuits

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Many electrical networks are designed to accept signals at certain points and to deliver a modified version of the signals. The usual arrangement is illustrated in Figure 1.

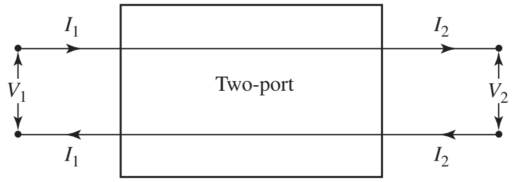

Figure 1 Electrical network

A current I_1 at voltage V_1 is delivered into a two-port, and it determines in some way the output current I_2 at voltage V_2. In practice, the relationship between the input and output currents and voltages is usually linear—they are related by a matrix equation:

$$\begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix}.$$

The coefficient matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is called the **transmission matrix** of the port. This matrix defines the two-port.

Figure 2 is an example of a two-port. The interior consists of a resistance R connected as shown. Let us show that the currents and voltages do, indeed, behave in a linear manner and determine the transmission matrix. Our approach will be to construct two equations: one expressing V_2 in terms of V_1 and I_1, and the other expressing I_2 in terms of V_1 and I_1. We will then combine these two equations into a single matrix equation.

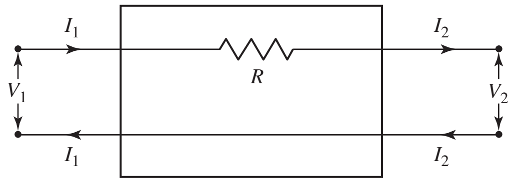

Figure 2 Two-port

We use the following law:

> ***Ohm's law:*** *The voltage drop across a resistance is equal to the current times the resistance.*

The voltage drop across the resistance is $V_1 - V_2$. The current through the resistance is I_1. Thus Ohm's law implies that $V_1 - V_2 = I_1R$. The current I_1 passes through the resistance R and exits as I_1. Thus $I_2 = I_1$. We write these two equations first in the standard form,

$$V_2 = V_1 - RI_1$$
$$I_2 = 0V_1 + I_1$$

and then as a single matrix equation,

$$\begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} 1 & -R \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix}.$$

The transmission matrix is $\begin{pmatrix} 1 & -R \\ 0 & 1 \end{pmatrix}$. Thus, if R is 2 ohms and the input voltage and current are $V_1 = 5$ volts and $I_1 = 1$ amp, respectively, we get

$$\begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

The output voltage and current are 3 volts and 1 amp, respectively.

In practice, a number of standard two-ports such as the one described above are placed in series to provide a desired voltage and current change. Consider the three two-ports of Figure 3, with transmission matrices **A**, **B**, and **C**.

Considering each port separately, we have

$$\begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = \mathbf{A}\begin{pmatrix} V_1 \\ I_1 \end{pmatrix}, \quad \begin{pmatrix} V_3 \\ I_3 \end{pmatrix} = \mathbf{B}\begin{pmatrix} V_2 \\ I_2 \end{pmatrix}, \quad \begin{pmatrix} V_4 \\ I_4 \end{pmatrix} = \mathbf{C}\begin{pmatrix} V_3 \\ I_3 \end{pmatrix}.$$

Substituting $\begin{pmatrix} V_2 \\ I_2 \end{pmatrix}$ from the first equation into the second gives

$$\begin{pmatrix} V_3 \\ I_3 \end{pmatrix} = \mathbf{BA}\begin{pmatrix} V_1 \\ I_1 \end{pmatrix}.$$

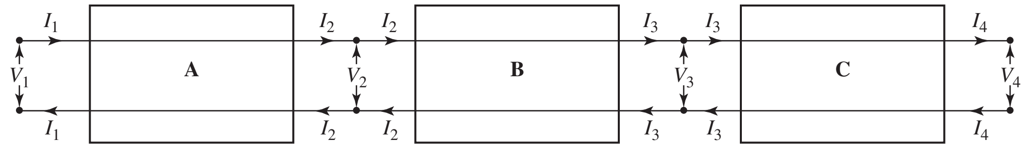

Figure 3 Two-ports in series

Substituting this last matrix $\begin{pmatrix} V_3 \\ I_3 \end{pmatrix}$ into the third equation gives

$$\begin{pmatrix} V_4 \\ I_4 \end{pmatrix} = \mathbf{CBA}\begin{pmatrix} V_1 \\ I_1 \end{pmatrix}.$$

Thus the three ports are equivalent to a single two-port. The transmission matrix of this two-port is the product **CBA** of the individual ports. Note that the placement of each port in the sequence is important because matrices are not commutative under multiplication.

Related Problems

In Problems 1–3, determine the transmission matrices of the two-ports in the given figure.

1. $V_1 = V_2$ because terminals are connected directly. Current through resistance R is $I_1 - I_2$. Drop in voltage across R is V_1.

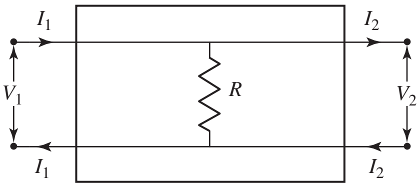

Figure 4 Two-port for Problem 1

2. Current through R_1 is $I_1 - I_2$. Drop in voltage across R_1 is V_1. Current through R_2 is I_2. Drop in voltage across R_2 is $V_1 - V_2$.

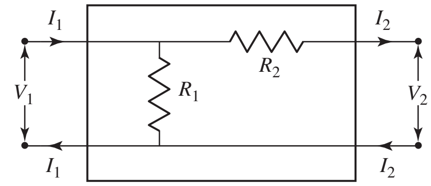

Figure 5 Two-port for Problem 2

3. Current through R_1 is I_1. Drop in voltage across R_1 is $V_1 - V_2$. Current through R_2 is $I_1 - I_2$. Drop in voltage across R_2 is V_2.

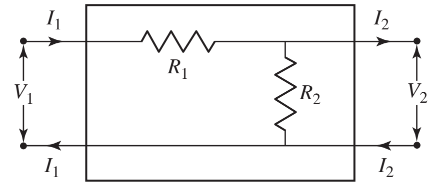

Figure 6 Two-port for Problem 3

4. The two-port in Figure 7 consists of three two-ports placed in series. The transmission matrices are indicated.

   (a) What is the transmission matrix of the composite two-port?

   (b) If the input voltage is 3 volts and current is 2 amps, determine the output voltage and current.

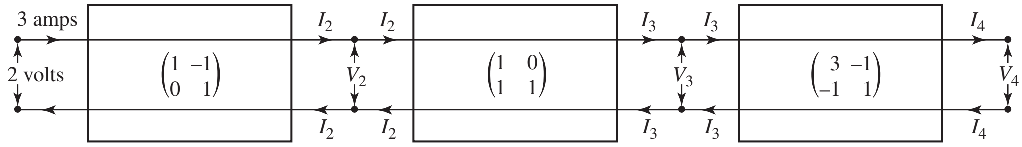

Figure 7 Two-ports in series for Problem 4