▸▸▸ PROJECT FOR SECTION 3.7

Road Mirages

Anton M. Jopko, Ph.D.
Department of Physics and Astronomy, McMaster University

All of us have driven down a highway on a sunny day and seen in the distance a shiny spot on the highway that looks like a patch of ice. This spot moves, disappearing and reappearing as we drive along.

The speed of light in a medium is given by $v = c/n$, where c is the speed of light in a vacuum, and n is the index of refraction for the medium. Because the speed of light cannot be greater than c, the index of refraction always satisfies $n \geq 1$. For cool air, the density and the index of refraction are larger so that the speed of light is slower. Conversely, for hot air, the density and index of refraction are smaller so that the speed of light is faster. When light travels between two media of different indices of refraction, it is bent or refracted. Figure 1 shows light being refracted by air as the density changes. The pavement is hot, and so is the air immediately above it. This set of circumstances makes the density of air smaller. Higher up the air is cooler, which makes the density of air increase with height; consequently its index of refraction increases. The index of refraction is nearly 1 at the surface of the road and increases very slowly with the height above the road. The bright light from the sky is refracted as it gets closer to the road, so that it enters the eyes of the driver as in Figure 1. In fact, the light never hits the road. Instead, it appears to come from directly on the road as a shiny patch in the distance.

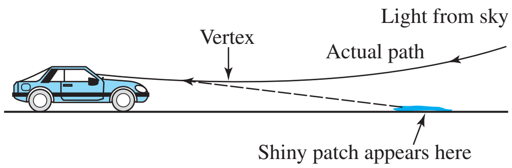

Figure 1 Refraction of light by air

Let us assume that the index of refraction of the air $n(y)$ depends only on the vertical height y above the road and that the x-axis is along the horizontal road. This implies that the path of light is symmetric about a vertical axis passing through the lowest point of the curve. We call this lowest point the **vertex**. If $y(x)$ denotes the equation of the path traveled by the light, it can be shown that y satisfies the nonlinear second-order differential equation

$$\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{1}{n}\frac{dn}{dy} \tag{1}$$

To solve this differential equation we need to know $n(y)$. We will consider a few examples of $n(y)$ in the *Related Problems* that follow. We will also see how to solve Equation (1) using the reduction of order technique. These cases may not be very realistic but they have the property that n is constant or increases with the height y. In either case, equation (1) is easily solved.

Related Problems

1. If the index of refraction $n(y)$ is an increasing function of y, explain why the path of light described by a solution $y(x)$ of (1) must be concave upward.
2. Suppose $n(y) = $ constant. This is a special case where the air is uniform in density. Equation (1) then becomes $d^2y/dx^2 = 0$.
   (a) What is the solution of this differential equation?
   (b) What is the concavity of the graph of this solution?
   (c) Why is the solution in part (a) physically reasonable for the given index of refraction?
3. (a) Suppose $n(y) = e^{y/a}$, where a and y are measured in meters and a is large (say, 10,000 m). Show that equation (1) becomes
   $$\frac{d^2y}{dx^2} = \frac{1}{a}\left[1 + \left(\frac{dy}{dx}\right)^2\right].$$
   (b) The dependent variable y is missing in the differential equation in part (a) and so the appropriate substitution is
   $$\frac{dy}{dx} = u \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{du}{dx}.$$
   Find the new differential equation for $u(x)$ and solve it.
   (c) Use your result from part (b) and the substitution $u = dy/dx$ to find the new differential equation for $y(x)$ and solve it.
   (d) Now suppose that the eyes of the driver are 1.2 m above the road, and the vertex of the path is 1.19 m above the road and 50 m in front of the driver. Use the solution in part (c) to find the distance from the car to the shiny patch on the road.
4. (a) Suppose $n(y) = \sqrt{y}$, $y \geq 1$, where y is measured in meters. Show that equation (1) becomes
   $$\frac{d^2y}{dx^2} = \frac{1}{2y}\left[1 + \left(\frac{dy}{dx}\right)^2\right].$$

(b) The independent variable x is missing in the differential equation in part (a) and so the appropriate substitution is

$$\frac{dy}{dx} = u \quad \text{and} \quad \frac{d^2y}{dx^2} = u\,\frac{du}{dy}.$$

Find the new differential equation for $u(y)$ and solve it.

(c) Use your result from part (b) and the substitution $u = dy/dx$ to find the new differential equation for $y(x)$ and solve it.

(d) Now suppose that the eyes of the driver are 1.2 m above the road, and the vertex of the path is 1.19 m above the road and 3 m in front of the driver. Use the solution in part (c) to find the distance from the car to the shiny patch on the road.