PROJECT FOR SECTION 3.11

Vibration Control: Vibration Absorbers

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Large structures such as buildings and bridges that are built from many parts cannot be made completely rigid. They move in response to natural disturbances, such as wind or seismic activity, and to man-made disturbances, such as the flow of traffic on a bridge. In particular, every structure has a set of special frequencies called natural frequencies, or resonance frequencies, at which it will respond particularly strongly. When subjected to periodic forces at one of these natural frequencies, a structure may respond with vibrations of an amplitude large enough to be uncomfortable for the occupants of the structure, or perhaps even dangerous to the structure itself. If engineers expect a structure to be subject to a periodic force at or near one of its natural frequencies, they may incorporate into its design a special device called a **tuned vibration absorber**. This is a device that suppresses vibration of the structure at one of its natural frequencies by transferring the energy that would cause such a vibration into vibration of a secondary mass. Tuned vibration absorbers are also used in smaller structures. For example, you might have seen small barbell shaped objects hanging from power transmission lines. These are tuned vibration absorbers called Stockbridge dampers that are installed to protect transmission lines from vibrations induced by wind. Tuned vibration absorbers can also be found in machinery of all sorts. (For example, most automobiles contain one or more tuned vibration absorbers.)

No matter how complicated a structure may be, we are concerned only with a particular one of its natural frequencies, so for our purposes a structure can be modelled as a spring/mass system whose resonance frequency represents the structure's troublesome natural frequency. Moreover, no matter how sophisticated the details of its design may be (and there are many variations based on such things as masses hanging from cables, masses attached to coil springs, masses suspended by pendulums, and even tanks of moving water) a tuned vibration absorber is in essence a spring/mass system attached to the structure. Therefore we can use the coupled spring/mass system in Figure 1(a) to model a structure (mass m_1 and spring constant k_1) equipped with a tuned vibration absorber (mass m_2 and spring constant k_2).

From Section 3.12, we already have a system of differential equations describing the motion of such a spring/

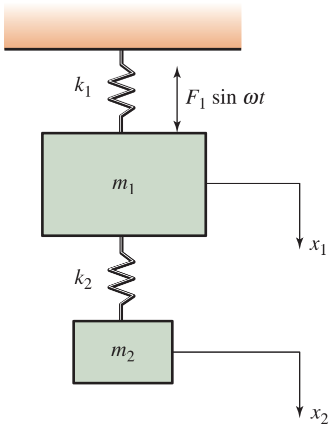

FIGURE 1(a)

mass system, to which we need only add a sinusoidal forcing term acting on m_1:

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1) + F_1 \sin \omega t \\ m_2 x_2'' &= -k_2(x_2 - x_1). \end{aligned} \tag{1}$$

We wish to investigate the behavior of this system as $t \to \infty$ for different values of the forcing frequency ω, and we claim that we can do this by finding a particular solution for the system, ignoring the general solution of the homogeneous version of the system. One might object that since there are no damping terms in the system, the general solution of the homogeneous version of the system has terms that persist as $t \to \infty$ for some initial conditions. However, if we were to add the slightest bit of damping to the system, the general solution of the homogeneous version of the resulting system would decay exponentially as $t \to \infty$. And in a real mechanical structure, there is always some damping. So in the context of modelling a real mechanical structure, we can regard a particular solution of the system as representing the behavior of any solution of the system as $t \to \infty$.

Based on our experience with the method of undetermined coefficients for a single second order equation, it makes sense to seek a particular solution of this system in which x_1 and x_2 are each of the form $c_1 \sin \omega t + c_2 \cos \omega t$. But since there are no first order derivatives, and the second derivative of a sine function is still a sine function, we can simplify matters and seek a particular solution in the form

$$x_1 = A \sin \omega t, \quad x_2 = B \sin \omega t \tag{2}$$

where A and B are constants to be determined.

1. By inserting the form indicated in (2) into (1), show that A and B satisfy the system of equations

$$\begin{aligned} (k_1 + k_2 - m_1 \omega^2)A - k_2 B &= F_1 \\ -k_2 A + (k_2 - m_2\omega^2)B &= 0 \end{aligned}$$

and that

$$A = \left(\frac{F_1}{k_1}\right)\left[\frac{1 - (\omega/\omega_2)^2}{\left(1 + \mu\left(\frac{\omega_2}{\omega_1}\right)^2 - \left(\frac{\omega}{\omega_1}\right)^2\right)\left(1 - \left(\frac{\omega}{\omega_2}\right)^2\right) - \mu\left(\frac{\omega_2}{\omega_1}\right)^2}\right]$$

where $\omega_1 = \sqrt{k_1/m_1}$ and $\omega_2 = \sqrt{k_2/m_2}$ are the natural frequencies of the structure alone and of the absorber alone, respectively, $\mu = m_2/m_1$ is the ratio of the mass of the absorber to the mass of the structure, and F_1/k_1 is the static displacement of m_1 under a constant force F_1.

Note that the mass m_1 will not vibrate at all when natural frequency ω_2 of the absorber equals the forcing frequency ω. So to eliminate vibrations of the structure at its natural frequency, we tune the absorber to the natural frequency of the structure by setting $\omega_2 = \omega_1$. It is convenient to measure the amplitude of the vibration of the structure with the quantity $\left|\frac{A}{F_1/k_1}\right|$, which is dimensionless and therefore independent of particular units of measurement:

$$\frac{A}{F_1/k_1} = \frac{1 - (\omega/\omega_1)^2}{\left(1 + \mu - \left(\frac{\omega}{\omega_1}\right)^2\right)\left(1 - \left(\frac{\omega}{\omega_1}\right)^2\right) - \mu}.$$

2. Use a graphing utility to graph $\left|\frac{A}{F_1/k_1}\right|$ as a function of $\frac{\omega}{\omega_1}$ for $\mu = 0.1, 0.2, 0.3$, and 0.4 on different sets of axes. (A viewing window of $0 \le \frac{\omega}{\omega_1} \le 2$ by $0 \le \left|\frac{A}{F_1/k_1}\right| \le 4$ will show the relevant features of these four graphs.)

You should see that the single natural frequency ω_1 of the structure has been replaced by two natural frequencies for the structure-absorber assembly. When the forcing frequency ω coincides with either of these two natural frequencies, the absorber makes things worse by inducing enormous vibrations of the structure. However, the absorber is effective for forcing frequencies near the natural frequency ω_1 of the structure.

3. Based on the graphs that you made above:
   (a) What happens to the two natural frequencies of the structure-absorber assembly as μ increases?
   (b) What happens to the range of the forcing frequency ω over which the absorber is effective?

To suppress the resonance of the structure-absorber assembly at its two natural frequencies, and to dissipate the energy that is transferred away from the structure into vibrations of the absorber mass m_2, we can incorporate damping into the absorber. This is represented in Figure 1(b) by a dashpot providing viscous damping with a damping constant of β. The resulting damped vibration absorber is sometimes called a **tuned mass damper**. To write its equations of motion, we add the appropriate damping terms to (1) and obtain:

$$m_1x_1'' = -k_1x_1 + k_2(x_2 - x_1) + \beta(x_2' - x_1') + F_1 \sin \omega t \quad (3)$$
$$m_2x_2'' = -k_2(x_2 - x_1) - \beta(x_2' - x_1').$$

A sinusoidal steady state solution can be derived for this system of the form

$$x_1 = A\sin(\omega t + \phi_1), \quad x_2 = B\sin(\omega t + \phi_2) \quad (4)$$

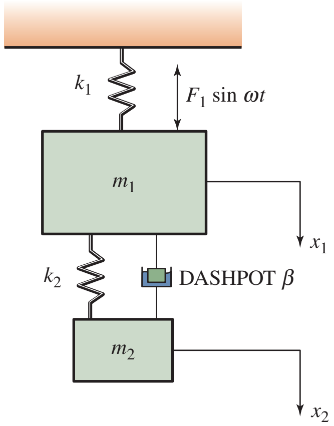

FIGURE 1(b)

where A and B are constants to be determined. We forgo this calculation here (because it is rather long and messy), but the result is that the amplitude A with which mass m_1 vibrates satisfies

$$\left|\frac{A}{F_1/k_1}\right| = \sqrt{\frac{\left(2\frac{\beta}{\beta_c}s\right)^2 + (s^2 - r^2)^2}{\left(2\frac{\beta}{\beta_c}s\right)^2(s^2 - 1 + \mu s^2)^2 + (\mu r^2 s^2 - (s^2 - 1)(s^2 - r^2))^2}} \quad (5)$$

where μ, ω_1, and ω_2 are as above, $r = \omega_2/\omega_1$, $\beta_c = 2m_2\omega_1$, and $s = \omega/\omega_1$.

4. (a) Verify that when $\beta = 0$, this expression reduces to the expression found in Problem 1.
   (b) Suppose that we tune the tuned mass damper as we did in undamped case ($r = 1$), and set $\mu = 0.1$. Use a graphing utility to graph $\left|\frac{A}{F_1/k_1}\right|$ as a function of s for $\beta/\beta_c = 1/20, 1/8$, and $1/2$ on the same set of axes. (A viewing window of $0.7 \le s \le 1.3$ by $0 \le \left|\frac{A}{F_1/k_1}\right| \le 18$ will show the relevant features of these three graphs.)
   (c) Describe how the behavior of the tuned mass damper changes as β/β_c increases from 0 to 1/2. (You may wish to include additional graphs for values of β/β_c between the three values given above.) Compare its performance to that of the undamped vibration absorber for small β/β_c and for large β/β_c, bearing in mind that there are two issues at stake: effective suppression of vibrations at (and near) the natural frequency ω_1 of the structure, and suppression of the potentially dangerous resonances of the structure-absorber assembly at its two natural frequencies.

You should notice the visually striking fact that there are two points through which the three graphs in Problem 4(b) all pass. It can be shown that for any fixed choice of r and μ, there are two points such that the graphs of $\left|\frac{A}{F_1/k_1}\right|$ vs. s pass through those two points for all values of β/β_c. The classical idea of "optimal" tuning for the tuned mass damper is to choose the parameters r and β/β_c so that these two special points share the same value of $\left|\frac{A}{F_1/k_1}\right|$, and so that this common value is (approximately) the maximum value of $\left|\frac{A}{F_1/k_1}\right|$.

5. The classical formulas given in many vibration engineering textbooks and handbooks for optimal tuning of the tuned mass damper are:

$$r = \frac{1}{1 + \mu}$$

and

$$\frac{\beta}{\beta_c} = \sqrt{\frac{3\mu}{8(1 + \mu)^3}}.$$

For a fixed value of $\mu = 0.1$:

(a) Use a graphing utility to plot the same three graphs that you plotted in 4(b) on the same set of axes, but with $r = 1/(1 + \mu)$ instead of $r = 1$. What does this special choice of r do?

(b) Now use a graphing utility to plot the single graph corresponding to $r = 1/(1 + \mu)$ and $\frac{\beta}{\beta_c} = \sqrt{\frac{3\mu}{8(1 + \mu)^3}}$. What does this special choice of $\frac{\beta}{\beta_c}$ do?

Finally we note that just as one can eliminate the dashpot from the tuned mass damper in Figure 1(b) to produce an undamped tuned vibration absorber based on the spring alone, as is analyzed in Problems 1–3, one can also eliminate the spring to produce a vibration absorber based on viscous damping alone called a **viscous vibration absorber**. Viscous vibration absorbers are less effective than tuned mass dampers, but are simpler to construct. Their performance can be analyzed by setting $k_2 = 0$ in (5).