PROJECT FOR SECTION 2.8

Tricky Timing: The Isochrones of Huygens and Leibniz

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Shortly before his death, Galileo Galilei (1564–1642) initiated the first attempt to design a pendulum driven clock. At that time, clocks were driven by falling weights or springs, and even the best of them gained or lost several minutes per day. Galileo hoped to achieve greater accuracy with a pendulum. In particular, Galileo believed that he had proven mathematically and verified experimentally that a pendulum is isochronous, that is, that the time it takes to complete one full swing is the same regardless of the size of the swing. Therefore, he reasoned, the frequency of oscillation of a pendulum is an especially reliable marker for the passing of time since it holds steady even if there are variations in the size of the pendulum's swing. Galileo did not live to complete his design, and though his son Vincenzio began to build a pendulum-driven clock, he too died in 1649 before he could finish it.

Practically speaking, Galileo's idea was timely; the pendulum-driven clock did turn out to be the next major step forward in time keeping. But his cherished belief in the isochronicity of the pendulum turned out to be in error. The great Dutch scientist and mathematician Christian Huygens (1629–1695) built the first pendulum-driven clock in 1656. It kept time to within one minute per day, and within two years Huygens and others produced even better models that kept time to within ten seconds per day. However, during this work careful measurements revealed that a pendulum is not in fact perfectly isochronous, though for small swings of just a few degrees it is very nearly so. The immediate implication of this discovery was simple: build clocks with pendulums that swing only a few degrees. But Huygens wanted to dig deeper, and began to pursue the question: how can perfect isochronicity be achieved if not with a simple pendulum?

To generalize the motion of the free end of a pendulum during half of one full swing, from its highest point to its lowest, consider Figure 1. There a bead is released from an initial position (x_0, y_0) at time $t = 0$ and slides down a wire in the xy-plane that is slick enough that friction is negligible, eventually reaching the origin (0, 0). In the case of a circular wire, the bead moves as if it were swinging on the end of a pendulum attached to the center of the circle. (Either the wire or the pendulum applies a force on the bead perpendicular to its direction of motion that constrains it to move in a semicircle.) To allow the bead to move in other ways, we simply consider wires of different

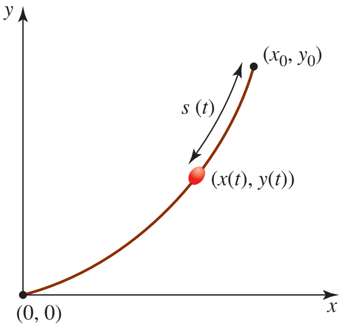

FIGURE 1

shapes connecting (x_0, y_0) with (0, 0). Then Huygen's question becomes: for what shape will the bead descend to the origin in the same time, regardless of its starting point (x_0, y_0)? In 1659, Huygens found the answer to this question in a remarkable tour de force of geometric physics without the aid of calculus or differential equations, neither of which yet existed. We can answer this question much more easily. From calculus, we need to know that if $s(t)$ represents the distance along the string that the bead has traveled at time t, the velocity ds/dt of the bead is given in terms of its horizontal and vertical velocities dx/dt and dy/dt by

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2. \tag{1}$$

And from physics, we need to know what Galileo himself discovered about the motion of a falling body released (with zero initial velocity) from a height y_0:

$$\left(\frac{ds}{dt}\right)^2 = 2g(y_0 - y), \tag{2}$$

where g is the acceleration of a freely falling object due to gravity.

Related Problems

1. **The Isochrone of Huygens**. Consider the arclength s indicated in Figure 1 as a function of y along the curve: $s = f(y)$. Then we can compute the time taken by a bead to descend from (x_0, y_0) to the origin using Equations (1) and (2) as follows:

$$T(y_0) = \int_{y_0}^{0} \frac{dt}{dy}\,dy = \int_{y_0}^{0} \frac{dt}{ds}\frac{ds}{dy}\,dy = \int_{y_0}^{0}\left(\frac{-1}{\sqrt{2g(y_0 - y)}}\right) f'(y)\,dy$$
$$= \int_0^{y_0} \frac{f'(y)}{\sqrt{2g(y_0 - y)}}\,dy.$$

(a) Use the change of variable $y = y_0 z$ to show that
$$T(y_0) = \frac{1}{\sqrt{2g}}\int_0^1 \frac{f'(zy_0)\sqrt{y_0}}{\sqrt{1 - z}}\,dz.$$

(b) We wish to find f such that T is a constant function of y_0. This will certainly be true if for $0 < z < 1$, $f'(zy_0)\sqrt{y_0}$ is a function only of z and not of y_0. We can ensure this by setting $\frac{\partial}{\partial y_0}(f'(zy_0)\sqrt{y_0}) = 0$. Show that this condition leads to the nonlinear differential equation $2f''(y)y + f'(y) = 0$ for $0 < y < y_0$.

(c) This last differential equation is first order in f'. Solve it to show that $f'(y) = c/\sqrt{y}$ for a constant $c > 0$.

(d) Dividing both sides of Equation (1) by $(dy/dt)^2$ yields $(ds/dy)^2 = (dx/dy)^2 + 1$. Use the result of (c) to show that

$$x = \int \sqrt{\frac{c - y}{y}}\, dy.$$

(e) Use the trigonometric substitution $y = c\sin^2\theta$, where $0 \le \theta \le \pi/2$, and the "half angle identities" $\sin^2\theta = (1 - \cos 2\theta)/2$ and $\cos^2\theta = (1 + \cos 2\theta)/2$, to show that the curve we seek can be parameterized by

$$x = \frac{c}{2}(\phi + \sin\phi) + k$$

$$y = \frac{c}{2}(1 - \cos\phi)$$

for some constant k and $0 \le \phi \le \pi$.

(f) Show that the curve given in (e) passes through $(x, y) = (0, 0)$ only when $k = 0$, and that in this case it passes through $(x, y) = (0, 0)$ when $\phi = 0$ and through (x_0, y_0) when ϕ_0 satisfies

$$x_0 = \frac{c}{2}(\phi_0 + \sin\phi_0)$$

$$y_0 = \frac{c}{2}(1 - \cos\phi_0).$$

(g) Show that this last system of equations can be solved for c and ϕ_0 in terms of x_0 and y_0 if and only if $0 < y_0/x_0 \le 2/\pi$. (So an isochrone connects (x_0, y_0) with the origin only when this condition holds.)

(h) The curve parameterized in (e) is sometimes called an **inverted cycloid**. When $k = 0$, it is traced out by a point on a circle of radius $c/2$ that is rolling along the line $y = c$ while hanging down below this line. Sketch a diagram of this, showing the position of the circle and the point for $\phi = 0, \pi/2$, and π. (To get started, look up the cycloid in any calculus book, or online.)

Huygens went on to design and build pendulum clocks modified to force the free end of the pendulum to follow the path of an inverted cycloid. But though these modifications inspired more great mathematics from Huygens, they were not of lasting value in the design of clocks since they turned out to introduce more problems than they solved. Huygens also went on to become the mathematical mentor of Gottfried Leibniz (1646–1716), who was later to become a founder of the calculus. Leibniz himself posed a different problem with a similar flavor in 1687: what shape should the wire in Figure 1 take so that the time it takes the bead to fall a given vertical distance is the same anywhere along the path of descent? This is equivalent to the vertical velocity of the bead being constant. Huygens was the first to respond to this challenge, guessing the solution and geometrically proving that it worked. Using differential equations, we can solve it systematically. We need the following generalization of Equation (2) that includes a non-zero initial velocity v_0 for the bead:

$$\left(\frac{ds}{dt}\right)^2 = 2g(y_0 - y) + v_0^2. \qquad (3)$$

2. **The Isochrone of Leibniz**. Let the constant vertical velocity dy/dt of the bead be $v < 0$. Then since $(dx/dt)^2 + (dy/dt)^2 = (dy/dt)^2[1 + (dx/dy)^2]$, we have $v_0^2 = v^2(1 + m^2)$, where $m = dx/dy$ at (x_0, y_0).

(a) Use Equations (1) and (3) to show that

$$\left(\frac{dx}{dy}\right)^2 = \frac{2g}{v^2}(y_0 - y) + m^2.$$

(b) Find the general solution of this differential equation.

(c) Show that requiring the solution curve to pass through (x_0, y_0) yields the particular solution

$$\frac{3g}{v^2}(x_0 - x) + m^3 = \left[\frac{2g}{v^2}(y_0 - y) + m^2\right]^{3/2}.$$

Under the change of variables $X = \frac{3g}{v^2}(x_0 - x) + m^3$, $Y = \frac{2g}{v^2}(y_0 - y) + m^2$, this equation becomes $X^2 = Y^3$, the graph of which is known as a **semicubical parabola**.

(d) Show that a bead departing (x_0, y_0) along the isochrone described in (c) reaches the origin when the following condition holds:

$$\left(\frac{3g}{v^2}x_0 + m^3\right)^2 = \left(\frac{2g}{v^2}y_0 + m^2\right)^3.$$

3. Modify the steps taken in parts (a)–(d) in Problem 2 to answer the following question: what shape should the wire in Figure 1 take so that the time it takes the bead to move a given horizontal distance is the same anywhere along the path of descent? This is equivalent to the horizontal velocity of the bead being constant.