PROJECT FOR SECTION **2.5**

Two Properties of the Sphere

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Despite its simple appearance, the sphere has numerous interesting geometric properties. Many of these properties are not visually apparent, but are revealed only by close mathematical examination. Here we use differential equations as a tool to study two such properties.

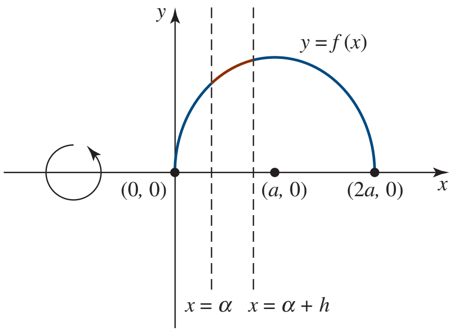

FIGURE 1

Consider a surface of revolution S formed by revolving the graph of a positive function $y = f(x)$ around the x-axis. (See Figure 1.) So that S is **smooth** (has no sharp edges), assume that f is differentiable. So that S is **closed** (encloses a region in 3-dimensional space), assume that the graph of f has exactly two x-intercepts. And so that S is **centrally symmetric** (has a center), assume that for some positive number a the graph of f is symmetric with respect to the vertical line $x = a$, so that its x-intercepts are located at (0, 0) and $(2a, 0)$ and the point $(a, 0)$ is the center of S.

Recall that the surface area of the slice of S lying between the planes $x = \alpha$ and $x = \alpha + h$ is

$$A = \int_{\alpha}^{\alpha+h} 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx. \tag{1}$$

If the graph of f is a semicircle:

$$y^2 + (x - a)^2 = a^2$$
$$y = f(x) = \sqrt{a^2 - (x - a)^2} \tag{2}$$

then S is a sphere of radius a. In this case, it is straightforward to calculate that $f(x)\sqrt{1 + [f'(x)]^2} = a$ so that $A = 2\pi ah$.

Note that A depends only on h (the width of the slice) and not on α (the location of the slice). This is a remarkable property of the sphere.

Property 1: The area of a slice of a sphere lying between two parallel planes depends only on the width of the slice and not on the location of the slice.

Do any other surfaces of revolution satisfy this property? Note that A is an integral over an interval of width h, not of f, but of the function $2\pi f(x)\sqrt{1 + [f'(x)]^2}$. If A depends only on h and not on α, then the area under the graph of $y = 2\pi f(x)\sqrt{1 + [f'(x)]^2}$ on the interval $[\alpha, \alpha + h]$ depends only on h (the width of the interval) and not on α (the location of the interval). This can only happen if $2\pi f(x)\sqrt{1 + [f'(x)]^2}$ is constant. (To see this, sketch the graph of a nonconstant function and imagine integrating it over different intervals of the same small width.)

1. Show that if $2\pi f(x)\sqrt{1 + [f'(x)]^2}$ is constant then for some constant c
$$[f(x)]^2 + [f(x)f'(x)]^2 = c^2. \tag{3}$$
Note that (3) is a nonlinear first order differential equation.
2. The appearance of $f(x)f'(x)$ suggests the substitution $u = [f(x)]^2$. Use this substitution to reduce (3) to a separable differential equation for x as a function of u, and find all the solutions of (3). (Careful! There is a singular solution that will not show up in the 1-parameter family of solutions to this separable equation. What is the geometric interpretation of this singular solution?)
3. What functions f in Figure 1 yield a smooth, closed, centrally symmetric surface of revolution S that satisfies Property 1?

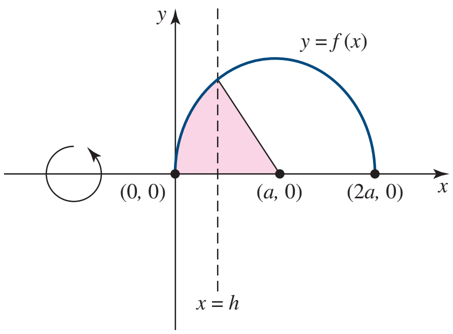

FIGURE 2

Now consider the ice cream cone shaped region produced by revolving the purple region in Figure 2 around the x-axis. It consists of a right circular cone, whose vertex is at $(a, 0)$ and whose base is at $x = h$, capped by the **end segment** of S sliced off by the plane $x = h$. The volume of this region is

$$V = \underbrace{\frac{1}{3}\pi[f(h)]^2(a - h)}_{\text{volume of cone}} + \underbrace{\int_0^h \pi[f(x)]^2\,dx}_{\text{volume of end segment}} \tag{4}$$

When S is a sphere this region is called a **spherical cone** and the end segment a **spherical segment** of height h. In this case a calculation using (2) in (4) reveals that $V = \frac{2}{3}\pi a^2 h$. This is another interesting property of the sphere.

Property 2: The volume of a spherical cone is proportional to the height h of the spherical segment forming the end of the spherical cone.

Do any other surfaces of revolution satisfy this property? That is, suppose S is a smooth, closed, centrally symmetric surface of revolution as in Figure 2 and the volume V given by (4) is proportional to h; must S be a sphere?

4. Show that if V is proportional to h, then for some constant c

$$\int_0^h [f(x)]^2\,dx + \frac{1}{3}[f(h)]^2(a - h) = ch. \tag{5}$$

5. Differentiate both sides of (5) with respect to h, remembering the second fundamental theorem of calculus, to obtain:

$$\frac{2}{3}[f(h)]^2 + \frac{2}{3}f(h)f'(h)(a - h) = c \quad \text{or, replacing } h \text{ with } x$$

$$[f(x)]^2 + f(x)f'(x)(a - x) = c \quad \text{(not the same “}c\text{”).} \tag{6}$$

6. Use the substitution $u = [f(x)]^2$ to reduce (6) to a separable differential equation for u as a function of x and find a 1-parameter family of solutions.
7. What functions f in Figure 2 yield a smooth, closed, centrally symmetric surface of revolution S that satisfies Property 2?

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