PROJECT FOR SECTION 2.2

When Differential Equations Invaded Geometry: Inverse Tangent Problems in the 17th Century

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The ancient Greeks classified geometric problems into three types. **Planar problems** were those requiring only lines and circles, which could be constructed in step-by-step fashion by straightedge and compass. **Solid problems** required the conic sections (ellipse, hyperbola, and parabola), which could be constructed only by intersecting cones and planes in three-dimensional space. **Linear problems** required even more exotic curves that could be constructed only by mechanical processes involving the juxtaposition of multiple simultaneous motions. Euclidean geometry provided a theory of planar problems that was considered rigorous because of the intuitive clarity and simplicity of straightedge and compass constructions. In contrast, a philosophical prejudice developed that solid problems, and especially linear problems, could never be treated as rigorously as planar problems because the construction of the curves involved was too abstract or too complicated, or both. Supporting this prejudice was the fact that certain problems involving solid and linear curves admitted no general method of solution at the time, though they could be solved in special cases by clever ad hoc arguments. Foremost among these difficult problems were the determination of tangent lines of curves, the calculation of areas bounded by curves (**quadrature**), and the calculation of the arclengths of curves (**rectification**).

The idea that curves beyond lines and circles could not be fully integrated into a rigorous theory of geometry persisted for centuries, but finally began to change in the 17th century, when Frenchmen Pierre de Fermat (1601–1665) and René Descartes (1596–1650), working independently, catalyzed a revolution in thought by applying the tools of algebra to the geometry of curves. This approach, which formed the basis of what eventually became analytic geometry, empowered them to formulate the first systematic methods for the determination of tangent lines. However, these methods were effective only for **algebraic curves**: curves that are graphs of polynomial equations in two variables. This class of curves includes lines, circles, the conic sections, and much more. Descartes was so excited about what he could do within this class of algebraic curves that he formulated his own theory of geometry based on it, complete with a method for geometrically constructing algebraic curves that was much more liberal that Euclid's ruler and compass method. But certain non-algebraic or **transcendental curves** were of increasing interest at the time.

One reason for the new interest in transcendental curves had to do with the advent of a new class of problems called **inverse tangent problems**: problems asking not to determine the tangent lines of a given curve, but rather to determine a curve whose tangent lines (or perhaps associated lines such as normal lines) satisfy some given property. In particular, it was discovered that a curve whose tangent lines satisfy some natural geometric condition might well turn out to be transcendental. This fact, along with the appearance of transcendental curves in mathematical physics, made it awkward to treat transcendental curves as isolated curiosities apart from the rest of the geometry. What was needed was a theory that would systematically address geometric problems associated with tangent lines, quadrature, and rectification involving algebraic and transcendental curves alike. Late in the 17th century, this theory was finally formulated. It was the calculus, which was pioneered independently in England by Isaac Newton (1643–1727) and in Germany by Gottfried Leibniz (1646–1716). Over time, calculus-based techniques for the solution of inverse tangent problems were developed and applied well beyond problems of elementary geometry, and were eventually consolidated into the systematic and efficient theory of first order differential equations that is presented in this chapter. Now even those inverse tangent problems that were cutting-edge challenges to the great mathematicians of the 17th century lie within the grasp of a beginning student of differential equations.

Related Problems

"I claim then that there is yet another analysis in geometry which is completely different from the analysis of Viète and of Descartes, who did not advance sufficiently in this, since its most important problems do not depend on the equations to which all of Descartes's geometry reduces. Despite what he had advanced too boldly in his geometry (namely, that all problems reduce to his equations and his curved lines), he himself was forced to recognize this defect in one of his letters; for de Beaune had proposed to him one of these strange but important problems of the inverse method of tangents, and he admitted that he did not yet see it clearly enough."—Gottfried Leibniz, personal letter, from *Philosophical Essays* by G. W. Leibniz, Roger Ariew, and Daniel Garber translators and editors, Hackett Publishing Company, Indianapolis, 1989.

Figure 1 illustrates the terminology associated with inverse tangent problems. We are given a curve and an axis, which we label the x-axis. Suppose that the tangent line of the curve at A intersects the x-axis at D, and the normal line to the curve at A intersects the x-axis at B. Then the **tangent** and **normal** of the curve at A are respectively defined to be the line segments $\overline{AD}$ and $\overline{AB}$. The **subtangent** and **subnormal** of the curve at A are respectively defined to be the projections of the tangent and normal at A onto the x-axis, that is, line segments $\overline{CD}$ and

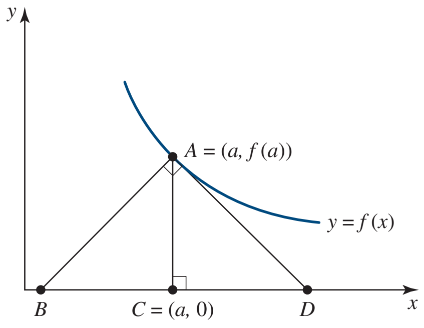

FIGURE 1 Inverse tangent problems project

$\overline{BC}$. (Of course, for a decreasing function, or a function whose graph is below the x-axis, the picture will look a bit different, but the definitions are the same.)

In 1638 the French nobleman Florimond Debeaune, a follower and friend of Descartes, posed to Descartes in a letter what is cited by historians of mathematics as the very first inverse tangent problem: to find a curve on which the ratio of the ordinate to the length of the subtangent is proportional to the difference between the ordinate and the abscissa. In his reply to Debeaune, Descartes devised both a graphical method of sketching these curves and a numerical method for calculating the coordinates of particular points on them. But he could not give analytical formulas for the curves, which he realized were not algebraic and therefore not within the purview of his general theory of geometry. Leibniz included a solution of this problem in his very first paper on his new theory of the calculus, which appeared in 1684, as an example to demonstrate the power of his methods.

1. Using Figure 1, show that Debeaune's problem translates into the differential equation $f'(a) = b(a - f(a))$, or $\dfrac{dy}{dx} = b(x - y)$ for some constant b.
2. Descartes recast this problem in terms of the new dependent variable $z = x - y - 1/b$. Show that with this change of variable, the differential equation in Problem 1 becomes $\dfrac{dz}{dx} = -bz$. Solve this differential equation to find z as a function of x. Then find y as a function of x on the **curves of Debeaune**. (There will be two constants in your formula: b and a constant of integration.)
3. Find and solve a differential equation for the functions f on whose graphs the length of the subtangent at $A = (a, f(a))$ is constant. Why is this problem often referred to as Debeaune's problem?

Some time between 1672 and 1676, when Leibniz was living in Paris, a prominent architect, physician, anatomist, and man of letters by the name of Claude Perrault posed to Leibniz a natural sister question to Debeaune's: which curves have tangents of constant length? Perrault showed Leibniz a vivid physical realization of such a curve by placing his pocket watch on a table, extending the watch's chain in a straight line so that it ended on an edge of the table, and pulling the end of the chain along that edge. The edge of the table then becomes our x-axis and, since at each moment the watch moves in the direction indicated by the chain, the chain itself is a tangent of constant length to the curve traced out by the watch. Leibniz did not publish his solution to this problem until 1693.

4. Derive and solve a differential equation for the function $y = f(x)$ whose graph solves Perrault's problem. (See Problem 28 in the Exercises of Section 1.3 and Problem 27 in the Review Exercises of Chapter 2. You will have to express x as a function of y. The solution looks a bit simpler if you assume that the curve passes through the point $(0, s)$, where s is the constant length of the tangent.)

A solution to Perrault's problem is known as a **tractrix**. Leibniz noticed that a machine based on Perrault's physical realization of the tractrix could, in effect, mechanically produce a graphical solution of the differential equation defining the tractrix. This thought inspired him to design a machine that could do the same for other differential equations. Such mechanical differential equation solvers based on "tractional motion" were designed and built by many mathematicians at various times right up until the advent of electronic computers. The tractrix has other engineering applications that have better stood the test of time, including applications in the design of mechanical objects such as bearings, gears, and valves, and in the design of audio speakers.

In Problems 5–7, derive a differential equation for the functions $y = f(x)$ whose graphs satisfy the given property. Then find all solutions of the resulting differential equation. (Be sure to account for any singular solutions.) Finally note that, in contrast to the famous problems above, the resulting curves would have been familiar to the ancient Greeks. They are all lines, circles, and conic sections with particular orientations and locations; describe them as such.

5. The length of the normal at $A = (a, f(a))$ is constant.
6. The length of the subnormal at $A = (a, f(a))$ is constant.
7. The length of the normal at $A = (a, f(a))$ is equal to the distance between $A = (a, f(a))$ and the origin $(0, 0)$.

Now do the same for Problem 8. Do the solution curves include any lines, circles, or conic sections? Are they all algebraic?

8. The length of the subtangent at $A = (a, f(a))$ is proportional to a.