▸▸▸ PROJECT FOR SECTION 16.2

Instabilities of Numerical Methods

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Finite-difference methods for numerical solutions of partial differential equations can be surprisingly inappropriate for numerical approximations. The main problem with finite-difference methods (especially with explicit iteration schemes) is that they may magnify the numerical round-off noise due to intrinsic instabilities. Such instabilities occur quite often in research work. An engineer should be prepared for this situation. After he or she spends many hours in development of a new numerical method for modeling of an applied problem and careful coding of the method with a computer language, the computer program could then turn out to be useless because of its dynamical instabilities.

Figure 1 illustrates a numerical solution of the heat equation with an explicit finite-difference method, where the time step k exceeds half of the squared step size h (see Example 1 in Section 16.2). It is expected that a solution of a heat equation for a rod of finite length with zero temperatures at the end points should exhibit a smooth decay of an initial heat distribution to the constant level of zero temperatures. However, the surface in Figure 1 shows that the expected smooth decay is destroyed by the noise that grows rapidly due to dynamical instabilities of the explicit method.

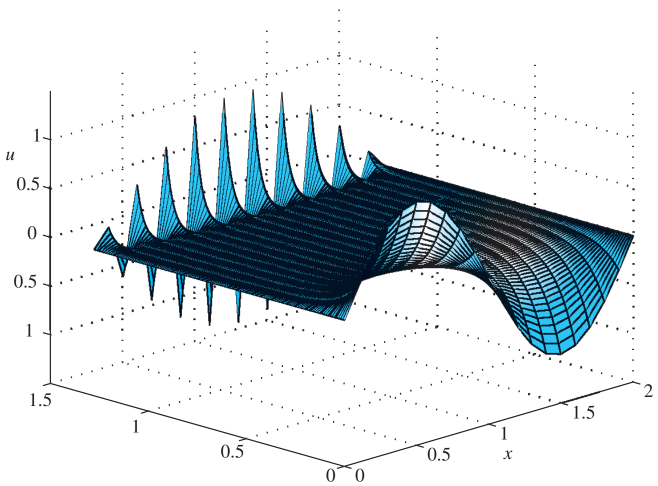

Figure 1 Numerical solution surface

Instabilities of numerical finite-difference methods can be understood with an elementary application of the discrete Fourier transform, which you may have studied in Section 15.5. The linear superposition principle and the discrete Fourier transform enable us to separate variables in a numerical finite-difference method and to study individual time evolution (iterations) of each Fourier mode of the numerical solution.

For simplicity, we shall consider the explicit finite-difference method for the heat equation $u_t = u_{xx}$ on the interval $0 \le x \le a$ subject to the zero boundary conditions at the end points $x = 0$ and $x = a$ and a nonzero initial condition at the time level $t = 0$. The numerical discretization leads to the explicit iteration scheme:

$$u_{i,\,j+1} = \lambda u_{i-1,\,j} + (1 - 2\lambda)u_{i,\,j} + \lambda u_{i+1,\,j} \tag{1}$$

where $u_{i,\,j}$ is a numerical approximation of the solution $u(x,\ t)$ at the grid point $x = x_i$ and the time level $t = t_j$, while $\lambda = k/h^2$ is the parameter of discretization. Let us freeze the time level $t = t_j$, $j \ge 0$ and expand the numerical vector $(u_{0,\,j},\ u_{1,\,j},\ \ldots,\ u_{n,\,j})$ defined on the equally spaced grid $x_i = ih$, $i = 0, 1, \ldots, n$, where $nh = a$, in the discrete Fourier sine-transform:

$$u_{i,\,j} = \sum_{l=1}^{n} a_{l,\,j} \sin\left(\frac{\pi i l}{n}\right), \quad i = 0,\ 1,\ \ldots,\ n \tag{2}$$

The boundary conditions $u_{0,\,j} = u_{n,\,j} = 0$ are satisfied for any $j \ge 0$. Due to the linear superposition principle, we shall consider each term of the sum in equation (2) separately. Hence we substitute $u_{i,\,j} = a_{l,\,j} \sin(\kappa_l i)$, $\kappa_l = \pi l/n$ into the explicit method (1) and obtain

$$a_{l,\,j+1} \sin(\kappa_l i) = (1 - 2\lambda)a_{l,\,j} \sin(\kappa_l i) + \lambda a_{l,\,j}\Big(\sin(\kappa_l(i+1)) + \sin(\kappa_l(i-1))\Big). \tag{3}$$

Using the trigonometric identity,

$$\sin(\kappa_l(i+1)) + \sin(\kappa_l(i-1)) = 2\cos(\kappa_l)\sin(\kappa_l i),$$

the factor $\sin(\kappa_l i)$ cancels out in equation (3), and we obtain a simple iteration formula for $a_{l,\,j}$:

$$a_{l,j+1} = Q_l a_{l,j},$$

where

$$Q_l = 1 - 2\lambda + 2\lambda\cos(\kappa_l) \tag{4}$$

Given that the factor Q_l is j-independent, it is clear the amplitude $a_{l,j}$ of the Fourier mode $\sin(\kappa_l i)$ changes in $j \ge 0$, according to the power of the factor Q_l:

$$a_{l,j} = Q_l^j a_{l,0}, \quad j \ge 0$$

The amplitude $a_{l,j}$ grows in j if $|Q_l| > 1$, and it is bounded or decaying if $|Q_l| \le 1$. Therefore, the stability of the explicit iteration method is defined from the constraint that

$$|Q_l| \leq 1, \text{ for all } l = 1, 2, \ldots, n \tag{5}$$

Because $Q_l < 1$ for $\lambda > 0$, the stability constraint (5) can be rewritten as

$$1 - 4\lambda \sin^2\left(\frac{\pi l}{2n}\right) \geq -1, \quad l = 1, 2, \ldots, n \tag{6}$$

which results in the *conditional stability* of the explicit method for $0 < \lambda \leq 0.5$. When $\lambda > 0.5$, the first unstable Fourier mode corresponds to $l = n$, which is responsible for a pattern of time growing space-alternative sequence of $u_{i,j}$. This pattern called is clearly seen in Figure 1.

Thus instabilities of finite-difference methods can be studied using the discrete Fourier transform, the linear superposition principle, and the explicit time-iteration factors. The same method can be applied to other finite-difference methods for heat and wave equations, and in general to a discretization of any linear partial differential equations with constant coefficients.

Related Problems

1. Consider the implicit Crank-Nicholson method for the heat equation $u_t = u_{xx}$ (see Example 2 in Section 16.2):

$$-u_{i-1,j+1} + \alpha u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} - \beta u_{i,j} + u_{i+1,j} \tag{7}$$

   where $\alpha = 2(1 + 1/\lambda)$, $\beta = 2(1 - 1/\lambda)$, and $\lambda = k/h^2$. Find the explicit formula for Q_l in Equation (4) and prove that the implicit Crank-Nicholson method (7) is *unconditionally stable* for any $\lambda > 0$.

2. Consider the explicit central-difference method for the heat equation $u_t = u_{xx}$:

$$u_{i,j+1} = 2\lambda(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + u_{i,j-1}. \tag{8}$$

   Using the same algorithm as in Problem 1, reduce Equation (8) to a two-step iteration scheme:

$$a_{l,j+1} = 4\lambda(\cos(\kappa_l) - 1)a_{l,j} + a_{l,j-1}. \tag{9}$$

   Using the explicit iteration scheme (4), find a quadratic equation for Q_l and solve it with the quadratic formula (see Example 1 in Section 11.2). Prove that the explicit central-difference method (8) is *unconditionally unstable* for any $\lambda > 0$.

3. Consider the explicit central-difference method for the wave equation $u_{tt} = c^2 u_{xx}$ (see Example 1 in Section 16.3):

$$u_{i,j+1} = \lambda^2 u_{i-1,j} + 2(1 - \lambda^2)u_{i,j} + \lambda^2 u_{i+1,j} - u_{i,j-1} \tag{10}$$

   where $\lambda = ck/h$ is the Courant number. Using the same algorithm as in Problem 2, find and solve the quadratic equation for Q_l. Prove that $|Q_l| = 1$ when both roots of the quadratic equation are complex. Prove that the stability constraint (5) is violated when both roots of the quadratic equation are distinct and real. Prove that the explicit central-difference method (10) is stable for $0 < \lambda^2 \leq 1$ and unstable for $\lambda^2 > 1$.

4. Consider the forward-time backward-space method for the transport equation $u_t + cu_x = 0$:

$$u_{i,j+1} = (1 - \lambda)u_{i,j} + \lambda u_{i-1,j} \tag{11}$$

   where $\lambda = ck/h$. Consider the complex discrete Fourier transform with the Fourier mode,

$$u_{i,j} = a_{l,j}e^{i\kappa_l i}, \quad \text{where } \kappa = \pi l/n, \ i = \sqrt{-1}$$

   and find the complex-valued factor Q_l in the one-step iteration scheme (4). Prove that the forward-time backward-space method (11) is stable for $0 < \lambda \leq 1$ and unstable for $\lambda > 1$.

5. Consider the backward-time central-space method for the transport equation $u_t + cu_x = 0$:

$$\lambda u_{i+1,j+1} + 2u_{i,j+1} - \lambda u_{i-1,j+1} = 2u_{i,j} \tag{12}$$

   Using the same algorithm as in Problem 4, prove that the backward-time central-space method (12) is *unconditionally stable* for any $\lambda > 0$.