PROJECT FOR SECTION **15.4**

The Uncertainty Inequality in Signal Processing

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Communication engineers interpret the Fourier transform as decomposing an information-carrying signal $f(x)$, where x represents time, into a superposition of pure sinusoidal "tones" having frequencies represented by a real variable. In fact, engineers usually think about the resulting "frequency domain" representation as much as or more than the "time domain" representation (that is, the signal itself)! A fundamental fact of signal processing is that the narrower a signal is in the time domain, the broader it is in the frequency domain. Conversely, the narrower a signal is in the frequency domain, the broader it is in the time domain. This effect is important because in practice, a signal must be sent in a limited interval of time and using a limited interval, or "band," of frequencies. In this project, we describe and investigate this tradeoff between duration and bandwidth both qualitatively and quantitatively. The results of our investigation will support a commonly quoted rule of thumb: The number of different signals that can be sent in a certain duration of time using a certain band of frequencies is proportional to the product of the time duration and the width of the frequency band.

Related Problems

We will use the complex form of the Fourier transform and inverse Fourier transform given in (5) and (6) of Section 15.4. We will use the notation $\hat{f}(\alpha)$ to denote the Fourier transform of a function $f(x)$ in a compact way that makes its dependence on f explicit—that is, $\hat{f}(\alpha) = \mathscr{F}\{f(x)\}$. We take f to be a real-valued function, and we warm up by noting two simple properties that $\hat{f}$ enjoys.

1. Show that if $\alpha > 0$, then $\hat{f}(-\alpha) = \overline{\hat{f}(\alpha)}$. So for any α, $|\hat{f}(-\alpha)| = |\hat{f}(\alpha)|$. (Here the notations $\bar{z}$ and $|z|$ represent the conjugate and the modulus of a complex number z, respectively.)
2. If k is a real number, let $f_k(x) = f(x - k)$. Show that

$$\hat{f}_k(\alpha) = e^{i\alpha k}\hat{f}(\alpha)$$

So shifting a signal in time does not affect the values of $|\hat{f}(\alpha)|$ in the frequency domain.

Keeping these facts in mind, we now consider the effect of narrowing or broadening a signal in the time domain by simple scaling of the time variable.

3. If c is a positive number, let $f_c(x) = f(cx)$. Show that

$$\hat{f}_c(\alpha) = \frac{1}{c}\hat{f}\left(\frac{\alpha}{c}\right).$$

Thus narrowing the signal function f in the time domain $(c > 1)$ broadens its transform in the frequency domain, and broadening the signal function f in the time domain $(c < 1)$ narrows its transform in the frequency domain.

To quantify the effect that we observe in Problem 3, we need to settle on a measure of the "width" of the graph of a function. The most commonly used measure is the root mean square width, which when applied to a signal f in the time and frequency domains yields a root mean square duration $D(f)$ and a root mean square bandwidth $B(f)$ given by

$$[D(f)]^2 = \frac{\displaystyle\int_{-\infty}^{\infty} x^2[f(x)]^2\,dx}{\displaystyle\int_{-\infty}^{\infty}[f(x)]^2\,dx}$$

and

$$[B(f)]^2 = \frac{\displaystyle\int_{-\infty}^{\infty} \alpha^2\,|\hat{f}(\alpha)|^2\,d\alpha}{\displaystyle\int_{-\infty}^{\infty}|\hat{f}(\alpha)|^2\,d\alpha}.$$

The bandwidth and duration are calculated relative to "centers" of $\alpha = 0$ and $x = 0$ because, by Problems 1 and 2, the graph of $|\hat{f}(\alpha)|^2$ is symmetric around $\alpha = 0$ in the frequency domain, and the signal can be shifted horizontally in the time domain without affecting the graph of $|\hat{f}(\alpha)|^2$ in the frequency domain.

4. Show that for a family of functions $f_c(x)$ defined in Problem 3, $D(f_c) \cdot B(f_c)$ is independent of c.
5. Show that for the family of functions $f_c(x) = e^{-c|x|}$, $D(f_c) \cdot B(f_c) = \dfrac{\sqrt{2}}{2}$. [*Hint*: By Problem 4, you can just take $f(x) = f_1(x)$. The necessary Fourier integral can be gleaned quickly from Example 3 of Section 15.3. To evaluate the integrals in $D(f)$ and $B(f)$, think about integration by parts and partial fractions, respectively.]

So the duration and the bandwidth of a signal are in a sense inversely proportional to each other under scaling of the time variable. What about the constant of proportionality? How small can $D(f) \cdot B(f)$ be? Remarkably, there is a lower limit for this product.

6. Derive the **uncertainty inequality**: If

$$\int_{-\infty}^{\infty}[f(x)]^2\,dx < \infty, \qquad \int_{-\infty}^{\infty}|\hat{f}(\alpha)|^2\,d\alpha < \infty,$$

and

$$\lim_{x\to\pm\infty}|x|[f(x)]^2 = 0,$$

then $D(f)\cdot B(f) \geq \frac{1}{2}$. Follow these steps.

(a) Establish Parseval's formula:

$$\int_{-\infty}^{\infty}[f(x)]^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(\alpha)|^2\,d\alpha.$$

[*Hint*: Apply the convolution theorem given in Problem 20, Exercises 15.4 with $g(x) = f(-x)$.

Specifically, apply the formula for the inverse Fourier transform given in (6) of Section 15.4, show that $\hat{g}(\alpha) = \overline{\hat{f}(\alpha)}$, and then let $x = 0$.]

(b) Establish the Schwartz inequality: For real-valued functions h_1 and h_2,

$$\left|\int_a^b h_1(s)h_2(s)\,ds\right|^2 \leq \left(\int_a^b [h_1(s)]^2\,ds\right)\left(\int_a^b [h_2(s)]^2\,ds\right)$$

with equality occurring only when $h_2 = ch_1$, where c is a constant [*Hint*: Write

$$\int_a^b [\lambda h_1(s) - h_2(s)]^2\,ds$$

as a quadratic expression $A\lambda^2 + B\lambda + C$ in the real variable λ. Note that the quadratic is nonnegative for all λ and consider the discriminant $B^2 - 4AC$.]

(c) Establish the uncertainty inequality. [*Hint*: First, apply the Schwartz inequality as follows:

$$\left|\int_{-\infty}^{\infty} xf(x)f'(x)\,dx\right|^2 \leq \left(\int_{-\infty}^{\infty}[xf(x)]^2 dx\right)\left(\int_{-\infty}^{\infty}[f'(x)]^2\,dx\right).$$

Use integration by parts to show that

$$\textstyle\int_{-\infty}^{\infty} xf(x)f'(x)\,dx = -\frac{1}{2}\int_{-\infty}^{\infty}[f(x)]^2\,dx.$$

Rewrite the second integral appearing on the right-hand side of the inequality using the operational property (11) of Section 15.4 and Parseval's formula.]

7. (a) Show that if f gives the minimum possible value of $D(f)\cdot B(f)$, then

$$f'(x) = cxf(x)$$

where c is some constant. Solve this differential equation to show that $f(x) = de^{cx^2/2}$ for $c < 0$ and $d =$ a constant. (Such a function is called a Gaussian function. Gaussian functions play an important role in probability theory.)

(b) Take the Fourier transform of both sides of the differential equation in part (a) to obtain a differential equation for $\hat{f}(\alpha)$ and show that $\hat{f}(\alpha) = \hat{f}(0)e^{\alpha^2/(2c)}$, where c is the same as in part (a). You will need the following fact:

$$\frac{d}{d\alpha}\hat{f}(\alpha) = \frac{d}{d\alpha}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial\alpha}f(x)e^{i\alpha x}\,dx$$

$$= \int_{-\infty}^{\infty} ixf(x)e^{i\alpha x}\,dx = \widehat{ixf(x)}.$$

(In Problem 35 in Exercises 9.11, we saw that $\int_{-\infty}^{\infty}e^{-x^2}dx = \sqrt{\pi}$. From this fact you can deduce that $\hat{f}(0) = \sqrt{2\pi/|c|}\cdot d$.)

So the minimum possible value of $D(f)\cdot B(f)$ is attained for a Gaussian function, whose Fourier transform is another Gaussian function!

The word "uncertainty" is associated with the inequality presented in Problem 6 because, from a more abstract point of view, it is mathematically analogous to the famous Heisenberg uncertainty principle of quantum mechanics. (The interpretation of this principle of quantum mechanics is a subtle matter, but it is commonly understood as "the more accurately one determines the position of a particle, the less accurately one knows its momentum, and vice versa.")