PROJECT FOR SECTION 15.4

Fraunhofer Diffraction by a Circular Aperture

Anton M. Jopko, Ph.D.
Department of Physics and Astronomy, McMaster University

The stars in the sky are an enormous distance from us, so they can be considered point sources of light. If you look at such a star through a telescope, you might expect to see just another point of light, albeit a much brighter one. However, this is not the case. Because it is a wave, light is diffracted as it passes through the circular aperture of the telescope so that the light is spread out over a small fuzzy region that we call the diffraction pattern. This project will investigate the shape of the diffraction pattern for light passing through a circular aperture of radius R.

For simplicity, we assume the light is of one wavelength λ, or color. This light has the form of a spherical wave front near the star, but by the time it reaches us the wave front forms a plane wave. All points on the wave front have the same phase. We now point the telescope with its circular aperture and lens directly at the star so that the plane wave fronts are incident from the left, as in Figure 1.

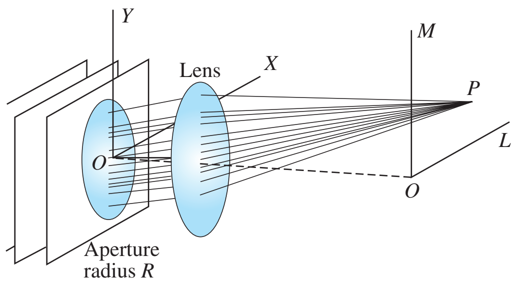

Figure 1 Diffraction of light

From Huygen's principle, each point in the opening of the circular aperture emits a wave in all directions. Fraunhofer diffraction requires that the waves leave the aperture in a nearly parallel bundle traveling toward a very distant point P. The only purpose of the lens is to form a point image of this parallel bundle at a much closer distance to the aperture. Diffraction would still occur even without the lens. The dotted line joining the two origins is also the axis of the aperture and lens. The LM system of coordinates is in the focal plane of the

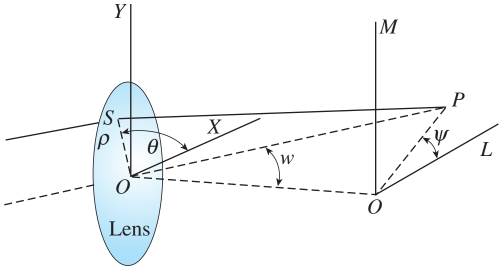

Figure 2

lens, and its origin is where all the light from the star would appear in the absence of diffraction. Because of diffraction, however, some light also appears at P. Point P is a general point but very close to O, being only arc seconds away.

In Figure 2, we have merged the aperture and the lens, because in practice the edge of the lens also defines the aperture. Because of the circular symmetry of the lens and the diffraction pattern, it is very desirable to convert to polar coordinates. Let a wave be emitted at a point S in the lens with coordinates (X, Y) or (ρ, θ) and arrive at P with coordinates (L, M) or angular coordinates (w, ψ). Then $X = \rho\cos\theta$, $Y = \rho\sin\theta$, and $L = w\cos\psi$, and $M = w\sin\psi$. Here ρ is the radial distance from the center of the lens to the source S of the emitted wave and θ is its polar angle; w is the **angular radius** of P and ψ is its polar angle.

The waves emitted at the aperture are in phase and have the same amplitude, but they all travel different distances to point P so they become out of phase there. The intensity of light at P will be proportional to the square of the resultant amplitude of all waves arriving there. We now need to calculate this resultant amplitude by taking into account the waves' phase differences.

We define the wave number of the incident and emitted waves to be $k = 2\pi/\lambda$. Then according to *Principles of Optics*, seventh edition, by Born and Wolf, the resultant amplitude at P from all the emitted waves in the aperture is just the Fourier transform of the aperture:

$$U(P) = C\iint_{aperture} e^{-ik(LX+MY)}\,dXdY$$

where C is a constant, proportional in part to the brightness of the star. The intensity at P will then be given by $|U(P)|^2$. This is the diffraction pattern for the star as a function of the angular radius w.

Related Problems

1. Show that the resultant amplitude at P using the two systems of polar coordinates can be written

$$U(P) = C\int_0^R\int_0^{2\pi} e^{-ik\rho w\cos(\theta-\psi)}\rho\,d\theta\,d\rho$$

2. Using the identity

$$\frac{i^{-n}}{2\pi}\int_0^{2\pi} e^{ix\cos\alpha}e^{in\alpha}\,d\alpha = J_n(x),$$

where J_n is the Bessel function of the first kind, show that the resultant amplitude reduces to

$$U(P) = 2\pi C\int_0^R J_0(k\rho w)\rho\ d\rho$$

for any ψ. We choose $\psi = 0$. (This expression is also known as a Hankel transform of a circular aperture.)

3. Using the recurrence relation

$$\frac{d}{du}\left[u^{n+1}J_{n+1}(u)\right] = u^{n+1}J_n(u),$$

show that

$$\int_0^x uJ_0(u)\,du = xJ_1(x)$$

4. Show that $U(P) = C\check{s}R^2\,\dfrac{2J_1(kRw)}{kRw}$. Therefore the intensity is given by

$$|U(P)|^2 = \left[\frac{2J_1(kRw)}{kRw}\right]^2 I_0$$

5. What is $\displaystyle\lim_{w\to 0}\frac{2J_1(kRw)}{kRw}$?

6. What is the physical significance of I_0?

7. What is the value of the smallest nonzero root of J_1? Using $\lambda = 550$ nm, $R = 10$ cm, and the smallest root just found, calculate the angular radius w (in arc seconds) of the central diffraction disk.

8. Draw a plot of $\dfrac{2J_1(kRw)}{kRw}$ as a function of kRw as well as the intensity, its square. The diffraction pattern of the star consists of a bright central disk surrounded by several thin, faint concentric rings. This disk is named the Airy disk in honor of G. B. Airy, who was the first to calculate the diffraction pattern of a circular aperture in 1826.

9. What happens to the angular width of the diffraction pattern if the radius R of the aperture is doubled?

10. What happens to the angular width of the diffraction pattern if the wavelength λ of the light is doubled?

11. What happens to the angular width of the diffraction pattern if the focal length of the lens is doubled?

12. Suppose that a circular aperture has the shape of an annulus with inner radius a and outer radius b. Find $U(P)$. (This result is of practical importance because reflecting telescopes almost always have an obstruction in the central part of the aperture.)

13. Suppose that the annulus in Problem 12 is very narrow such that $b = a + \Delta a$, with Δa being small but not infinitesimal. Show then that the *approximate* resultant amplitude is given by $U(P) = C(2\pi a\Delta a)J_0(kwa)$. [*Hint:* Interpret the result $U(P)$ from Problem 12 as an approximation for $\dfrac{d(uJ_1(u))}{du} = uJ_0(u)$ with $u = kwa$.]