PROJECT FOR SECTION 13.4

Making Waves: Convection, Diffusion, and Traffic Flow

Jeff Dodd, PhD
Mathematical, Computing, and Information Sciences Department
Jacksonville State University

The heat equation and wave equation are derived in Section 13.2 to describe very different phenomena. And indeed their solutions (studied in Sections 13.3 and 13.4 respectively) behave very differently. However, many important differential equations involve both heat-like and wave-like terms. Here we study an example of this in the context of traffic flow on a highway, some aspects of which are modelled by traffic engineers using differential equations.

We idealize a one-way highway as the x-axis and let t represent time. We represent the distribution of cars on the highway by a **density function** $u(x, t)$ giving the density of cars (cars per unit length) at a position x and time t. We represent the flow of traffic by a **flux function** $\phi(x, t)$ giving the number of cars passing position x per unit time at time t. For the sake of simplicity, we assume that cars do not enter or leave the highway. Thus on any interval $a \le x \le b$, the rate of change of the number of cars in the interval is equal to the number of cars entering the interval at $x = a$ minus the number of cars leaving the interval at $x = b$:

$$\frac{d}{dt}\int_a^b u(x, t)\, dx = \phi(a, t) - \phi(b, t).$$

This relation can be converted to a differential equation, since at any time t and on any interval $a \le x \le b$,

$$\int_a^b \frac{\partial u}{\partial t}(x, t)\, dx = -\int_a^b \frac{\partial \phi}{\partial x}(x, t)\, dx$$

$$\int_a^b \left(\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x}\right) dx = 0 \qquad (1)$$

from which it follows that at any time t and any position x,

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = 0. \qquad (2)$$

The equation (2) is the **general conservation law** governing the flow of traffic under the assumption that the total amount of traffic is conserved. Further modelling assumptions can be incorporated into the form of the flux function ϕ, which can depend on x and t directly or through u and the derivatives of u.

If all the cars move together at the same velocity, then $\phi = cu$ where c is that velocity, $c > 0$ indicating motion in the positive x direction, and $c < 0$ indicating motion in the negative x direction. The resulting equation is known as the **convection equation**:

$$u_t + cu_x = 0. \qquad (3)$$

1. Use the methods of Exercise 12 of Section 13.4 to show that the general solution to the convection equation is $u(x, t) = f(x - ct)$ where $u(x, 0) = f(x)$ is the initial traffic density. Thus every solution of the convection equation is a traveling wave solution of the wave equation: a wave that maintains its shape while moving with a constant velocity.

Next suppose that, perhaps during a traffic jam, traffic moves according to the flux function $\phi = -ku_x$ for some $k > 0$, that is, traffic moves from high density regions to low density regions at a rate proportional to the negative of the density gradient. The resulting equation is known both as the heat equation and as the **diffusion equation**:

$$u_t - ku_{xx} = 0. \qquad (4)$$

Thus both the heat equation and the convection equation (which by Problem 1 is essentially a one-way wave equation) are conservation laws.

While it is hard to imagine traffic actually moving according to the diffusion equation, even in a terrible traffic jam, it is easy to imagine drivers tending to seek out lower traffic density while also moving toward their destination. This can be modelled by the flux function $\phi = cu - ku_x$ where $k > 0$, resulting in the **diffusion–convection equation**:

$$u_t + cu_x = ku_{xx}. \qquad (5)$$

2. Show that if $u(x, t)$ satisfies (5) then the function $u(\eta, t)$ where $\eta = x - ct$ satisfies the diffusion equation $u_t(\eta, t) = ku_{\eta\eta}(\eta, t)$. So the processes of convection and diffusion proceed independently in (5).

More realistic traffic flow models reflect the fact as the traffic density increases cars move more slowly, and at a certain critical density D cars cease to move at all. So let's represent the car velocity $v(u)$ by a velocity function $v(u)$ such that $v'(u) < 0$ and $v''(u) \le 0$ for $0 < u < D$ as in Figure 1. The

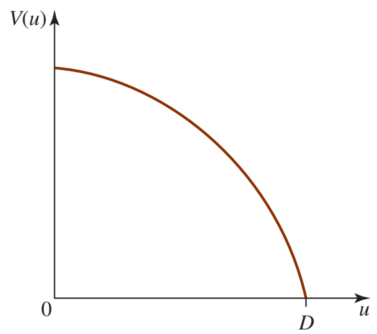

FIGURE 1 Car velocity as a function of traffic density.

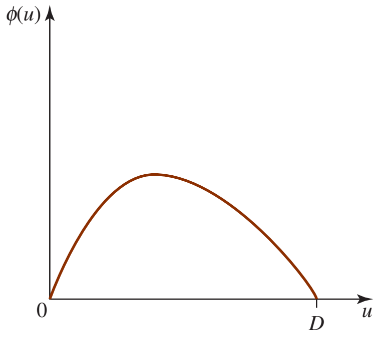

FIGURE 2 Traffic flux as a function of traffic density.

flux function based on v is $\phi(u) = uv(u)$ as in Figure 2 and the resulting **nonlinear convection equation** is

$$u_t + \phi'(u)u_x = 0. \tag{6}$$

Incorporating a diffusion effect into (6) yields the nonlinear diffusion–convection equation known as the **generalized Burgers equation**:

$$u_t + \phi'(u)u_x = ku_{xx}. \tag{7}$$

3. Check directly (by differentiating) that under our assumptions on $v(u)$, $\phi''(u) < 0$ for $0 < u < D$ so that $\phi'(u)$ is a decreasing function of u for $0 \le u \le D$.
4. By inserting the form $u = f(x - ct)$ into (6), find an ordinary differential equation for f and show that the only traveling wave solutions of (6) are constant.

Here we seek to model the following situation: low density traffic moving forward with high speed encountering high density traffic moving forward with slow speed. We have all seen what actually happens: a transition zone between low density and high density traffic forms and moves much as a traveling wave moves. But from Problem 4 we know that there are no nontrivial traveling wave solutions of (6).

Worse still, it can be shown that with a flux ϕ such as we have specified, any initial data $u(x, 0)$ for (6) that puts a region of lower density traffic behind a region of higher density traffic leads to a solution of (6) that develops a discontinuity after some finite time. Essentially, the low density traffic moves faster than the high density traffic and eventually catches up to it. If the low density, high speed drivers do not slow down until the traffic density at their current position becomes high as in (6) an abrupt (that is, discontinuous) change in traffic density and speed will develop. We have all experienced this when we have been surprised by an unforseen increase in traffic density, and attendant decrease in traffic speed, and have had to jam on our brakes to try to avoid a crash. In other words, if low density traffic follows high density traffic, the dependence of traffic velocity on traffic density in (6) given by a flux ϕ such as we have specified will increase the gradient u_x of the traffic density u to the point where u forms a jump discontinuity.

Of course, drivers do in fact attempt to adjust their speeds not only according to the traffic density at their current positions but also according to the gradient of the traffic density (i.e., according to what they see some distance ahead of them). In order to avoid abrupt changes in traffic density, they try to remain spread out by slowing down early when they see that the traffic density ahead of them is increasing. This effect is modelled by the diffusion term in (7).

So we seek a traveling wave solution $u = f(x - ct)$ of (7) satisfying the following conditions for traffic densities u_1 and u_2 with $0 \le u_1 < u_2 \le D$:

$$\lim_{\eta\to-\infty} f(\eta) = u_1, \quad \lim_{\eta\to\infty} f(\eta) = u_2, \quad \text{and} \quad \lim_{\eta\to\pm\infty} f'(\eta) = 0. \tag{8}$$

5. Insert the form $u = f(x - ct)$ into (7) to obtain a second order ordinary differential equation for f, and show that antidifferentiating this equation once yields

$$\frac{df}{d\eta} = \frac{1}{k}[\phi(f) - (cf + I)] \tag{9}$$

where I is a constant of integration.

6. By letting $\eta \to \pm\infty$, show that in order for (8) to hold, I and c must satisfy

$$\phi(u_2) = cu_2 + I \quad \text{and} \quad \phi(u_1) = cu_1 + I. \tag{10}$$

7. Use a sketch based on Figure 2 to explain why, for $0 \le u_1 < u_2 \le D$, it is always possible to choose I and c so that (10) is satisfied.
8. Now fix u_1 and u_2 such that $0 \le u_1 < u_2 \le D$ and let I and c in (9) satisfy (10).

   (a) Using your sketch from Problem 7, sketch a graph of $y = \frac{1}{k}[\phi(f) - (cf + I)]$ and a phase line for the autonomous differential equation (9). (See Section 2.1.2.)

   (b) Sketch a graph of $y = f(\eta)$ for a solution of (9). (How many inflection points does the graph have, and why?)

   (c) If the solution to (9) in the special case when $k = 1$ is denoted $F(\eta)$, show that $f(\eta) = F(\eta/k)$ satisfies (9), so that for any value of k the desired traveling wave solution to (7) is

$$u = F\left(\frac{x - ct}{k}\right). \tag{11}$$

   (d) Show that for any value of k,

$$c = \frac{\phi(u_2) - \phi(u_1)}{u_2 - u_1}. \tag{12}$$

   (e) Use your sketch from Problem 7 to show that c can be positive, negative, or zero. Describe what is happening on the highway in each of these cases.

9. Let $v(u) = 4 - 2u$ and $D = 2$ so that $\phi(u) = 4u - 2u^2$ for $0 \le u \le 2$. Solve (9) explicitly in the special case when $u_1 = 1$ and $u_2 = 2$. Then set $f(0) = 3/2$ and use a graphing utility to graph the resulting function f for $k = 10$, 1, and 1/10.

The effect of diffusion is made remarkably explicit in (11) and should also be clear in your graphs from Problem 9. The larger the value of k, the less abrupt the transition from low density to high density. On the other hand as $k \to 0$, transforming (7) into (6), the traveling waves (11) maintain the same velocity c, but approach the function

$$u(x, t) = \begin{cases} u_1 & \text{for } x - ct < 0 \\ u_2 & \text{for } x - ct > 0. \end{cases}$$

This function has a jump discontinuity that propagates with velocity c. It is not differentiable where $x = ct$, so it cannot satisfy the differential conservation law (6) when $x = ct$. But it can be shown that it does satisfy the corresponding integral conservation law (1) on any interval $a \leq x \leq b$. It is an example of a **shock wave solution** of (6): a solution that satisfies (1) on any interval $a \leq x \leq b$ and satisfies (6) everywhere except along a certain curve, called a **shock path** in the $x - t$ plane, where it is discontinuous. The smooth traveling waves (11) that approach the shock wave solution as $k \to 0$ are called **shock profiles**.

The nonlinear diffusion–convection equation (7) has smooth shock profiles modelling transitions from low traffic density to high traffic density because the sharpening effect of nonlinear convection on the density function is balanced by the smoothing effect of diffusion. The only solutions of the nonlinear convection equation (6) modelling such transitions are discontinuous shock wave solutions. In general, shock wave solutions of nonlinear conservation laws are usually considered physically significant only when they are limits of smooth shock profiles obtained by balancing nonlinear convection with diffusion because, in reality, some diffusive effect is nearly always present even if it is very small. For example, in the nonlinear equations governing the velocity of a flowing fluid, the diffusive effect is due to the conversion of kinetic energy to heat by internal friction, a process called **dissipation** that is incorporated into the equations using a property of the fluid called **viscosity**. This is why, even in other contexts, a diffusion term is sometimes called a **dissipation term** or **viscosity term** and the shock wave limit of shock profile solutions is sometimes called a vanishing viscosity limit.