## *Application Essay*

# Steady Transonic Flow Past Thin Airfoils

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When air is moving around a body, the speed of the air close to the body surface is different from that far away from the body. The main affecting parameter in this phenomenon is the geometry of the body. When the free stream speed (speed of air far away from the body) is close to the speed of sound (around 340 m/s), the flow is named transonic flow. In transonic flow, the local speed of air (speed on the surface of the body) is smaller than the speed of sound in the subsonic zone, greater than the speed of sound in the supersonic zone, and equal to the speed of sound in the sonic zone. For example, the flow on a thin circular arc airfoil, as shown in Figure 1, is accelerated near the leading edge of the airfoil until it reaches the speed of sound (sonic line). The flow continues to accelerate to supersonic speeds on the middle part of the airfoil. Then the supersonic zone is terminated by a shock wave near the trailing edge of the airfoil.

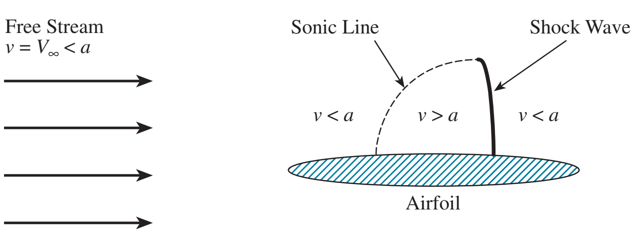


**Figure 1** Transonic Flow Past an Airfoil

The basic principles of this problem are mass conservation, energy conservation, and momentum conservation. Starting from the basic principles and utilizing simplifying assumptions and geometrical transformations, one can simulate the problem mathematically by the following partial differential equation (PDE):

$$[K - (\gamma + 1)M_\infty^2 \phi_x]\phi_{xx} + \phi_{yy} = 0$$

where $K = (1 - M_\infty^2)/\delta^{2/3}$ is the transonic similarity parameter, $\delta$ is the airfoil thickness ratio, $M_\infty = v/a$ is the Mach number, $\gamma$ is the specific heat ratio (1.4 for air), and $\phi$ is the perturbation velocity potential (the differentiation of it at any direction gives the change of speed of air due to the existence of the body at this direction).

This is a second-order nonlinear PDE, and in order for the problem to be well posted, we need two boundary conditions:

(1) *The far-field boundary condition:* The flow far away from the body does not feel the existence of the body. Accordingly, the perturbations are zero in all directions $\phi_x = \phi_y = 0$ when $x$ or $y$ tends to infinity.
(2) *The no-penetration boundary condition:* The flow cannot penetrate the surface of the body. Accordingly, the perturbation normal to the surface $\phi_n$ is zero on the surface. For thin airfoils, this condition can be approximated by $\phi_x(x, 0) = F'(x)$, where $y = F(x)$ on the surface of the body.

Defining $A = K - (\gamma + 1)M_\infty^2\phi_x$, the governing equation will be an elliptic, parabolic, or hyperbolic PDE according to the sign of $A$. When $A < 0$ the equation is hyperbolic, when $A > 0$ the equation is elliptic, and when $A = 0$ the equation is parabolic. It happens in this problem that the sign of $A$ changes at different zones. That is, the characteristics of the governing equation change according to the value of the local flow speed $\phi_x$.

One more simplifying transformation $\phi = (\Phi + Kx)/(\gamma + 1)$ can be used to put the governing equation in a form suitable for computations:

$$-\Phi_x\Phi_{xx} + \Phi_{yy} = 0.$$

## Numerical Solution

The finite-differencing technique is used to solve the above problem. Since the governing equation is a mixed-type PDE, different differencing schemes were chosen at each zone to pass the information in the right direction as shown in Figure 2. The resulting algebraic system of equations is solved iteratively because of the nonlinear nature of the governing equation.

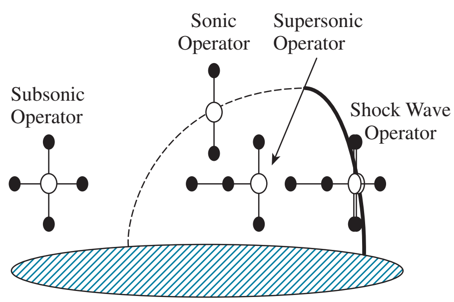


**Figure 2** Finite-Differencing Scheme

## ■ Finite-Differencing Scheme

For the elliptic PDE there are no real characteristic lines. Accordingly, a disturbance at any point will affect the rest of the elliptic domain. For the hyperbolic PDE there are two real characteristic lines. Accordingly, a disturbance at a point will affect the part of the domain impeded between the characteristic lines (zone of action). For the parabolic PDE there is only one real characteristic line. Accordingly, a disturbance at any point will propagate along that line.

### *Subsonic Operator*

Using central differencing in both the $x$ and the $y$ direction:

$$-\{(\Phi_{i+1,j} - \Phi_{i-1,j})/2\Delta x\}[(\Phi_{i+1,j} - 2\Phi_{i,j} + \Phi_{i-1,j})/\Delta x^2] + [(\Phi_{i,j+1} - 2\Phi_{i,j} + \Phi_{i,j-1})/\Delta y^2] = 0.$$

This is a stable elliptic operator when the first term $\Phi_c = \{(\Phi_{i+1,j} - \Phi_{i-1,j})/2\Delta x\}$ is negative.

### *Supersonic Operator*

Using backward differencing in the $x$ direction and central differencing in the $y$ direction:

$$-\{(\Phi_{i,j} - \Phi_{i-2,j})/2\Delta x\}[(\Phi_{i,j} - 2\Phi_{i-1,j} + \Phi_{i-2,j})/\Delta x^2] + [(\Phi_{i,j+1} - 2\Phi_{i,j} + \Phi_{i,j-1})/\Delta y^2] = 0.$$

This is a stable hyperbolic operator when the first term $\Phi_b = \{(\Phi_{i,j} - \Phi_{i-2,j})/2\Delta x\}$ is positive.

### *Sonic Operator*

There will be one point on $j$ = constant line on which neither of the previous operators is locally stable ($\Phi_b < 0$, $\Phi_c > 0$). Thus a parabolic operator is introduced by setting $\Phi_x = 0$:

$$[(\Phi_{i,j+1} - 2\Phi_{i,j} + \Phi_{i,j-1})/\Delta y^2] = 0.$$

### *Shock Wave Operator*

At any point on the shock wave surface there is a strong discontinuity in the flow parameters. The flow near its inner surface is supersonic ($\Phi_b > 0$), and on the outer surface the flow is subsonic ($\Phi_c < 0$). Accordingly, the shock wave operator is obtained by adding the subsonic and the supersonic operators.

## ■ Algebraic System of Equations

Two points should be elaborated here. First, the governing equation is nonlinear. Usually, that requires iterations to acquire a solution. Second, the chosen operators use three points at most (e.g., $i$, $i + 1$, $i - 1$). So the system of equations is expected to be tridiagonal.

### *Line Successive Over Relaxation (LSOR)*

Let $\Phi = \{\Phi_1, \Phi_2, \ldots, \Phi_{\text{Max}}\}^T$ be a vector of $i$ = constant line. The following LSOR formula was used to accelerate the convergence:

$$\Phi_i^{n+1} = \omega\Phi_i^{n+1} + (1 - \omega)\Phi_i^n,$$

where $\omega$ is a relaxation parameter. For the present problem, it is chosen to be over relaxation for the subsonic point $\omega = 1.2$ and under relaxation for the supersonic point $\omega = 0.4$, and $n$ is the iteration number for the entire flow field. Iterations are performed on successive vertical lines and by sweeping the flow field along the flow direction which is the maximum gradient direction.

### *Tridiagonal System*

For each line $i$ = constant the equation is formed as:

$$[A_{i,j}]\{\Phi_i\} = \{f_i\},$$

where $A$ is a tridiagonal matrix. This equation is solved using a tridiagonal solve from the United States Internet Mathematical Library called SGTSL. This subroutine requires the three diagonals and the constant vector $f_i$ to be sent in four vectors ($c,d,e,b$) and returns with the solution in vector $b$.

## Sample Results

Sample results for a circular arc airfoil with thickness ratio $\delta = 0.6$ at different values of the free stream Mach number are presented in this section. Results are presented in the form of coefficient of pressure $Cp \cup -2\Phi_x$ against

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the distance from the leading edge of the airfoil. As shown in Figure 3, the supersonic pocket starts to build up at $M = 0.8$, and the shock wave discontinuity becomes very clear at $M = 0.84$.

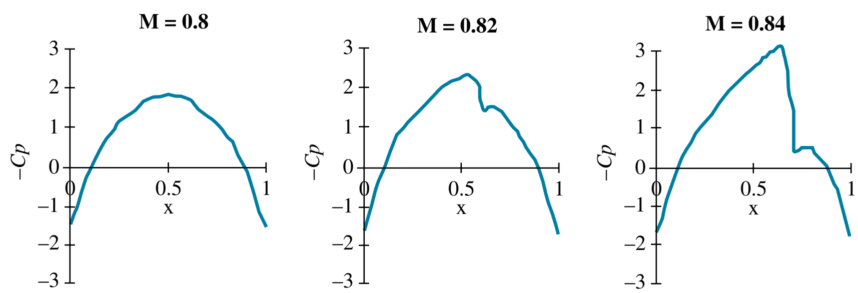


**Figure 3** Variation of Pressure Coefficient with Distance from the Airfoil Leading Edge

## Reference

[1] Ali, S. F., "*Steady Transonic Flow Past a Thin Airfoil,*" Eighth International Conference on Applied Mechanics and Mechanical Engineering, Military Technical College, Cairo, Egypt, May 12–14, 1998.

**Sherif F. Ali** received his B.S. and M.S. degrees, both in aeronautical engineering, from the Military Technical College in Cairo, Egypt, and his Ph.D. in aerospace technology from Arizona State University. Dr. Ali has over 10 years of research, teaching, and hands-on wind tunnel experience. His specialties include aerodynamics, aero-acoustics, fluid dynamics, and dynamics of flight, complemented by an extensive computer programming and numerical analysis background. He currently teaches at the University of Arizona and in Cairo, Egypt.