Application Essay

Atmospheric Drag and the Decay of Satellite Orbits

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Ever since the first Sputnik was launched into space, there have been artificial satellites circling the earth. These range from small pieces of space junk to large objects like the Hubble telescope and manned space stations. For most of their lives the main force acting on the objects is the earth's gravitational field. However, drag caused by the atmosphere will cause the orbit of any satellite to decay slowly, and, left alone, the satellite will ultimately fall to earth.

Small objects burn up in the earth's atmosphere and do not reach the earth's surface. Large objects have internal propulsion systems that can be used to maintain their orbits. However, in 1979 the internal propulsion system of one large object, Skylab, failed, and it entered the earth's atmosphere. Skylab was large enough that some small pieces survived and landed. They were very hot.

An exact mathematical model of the final few revolutions of the decaying orbit of a satellite is very complicated and difficult to obtain. In order to get an approximation we can solve, we must make many simplifying assumptions. Two common assumptions are that the earth is a perfect sphere and that the motion of the object is essentially two-dimensional. We will now derive a model for the motion of a decaying satellite.

The motion of the object is governed by Newton's second law: $F = ma$. In this analysis, we will assume that the only forces acting on the object are the gravitational attraction of the earth (ignoring the effects of the moon and sun and other bodies) and drag due to the atmosphere. Let the center of the earth be the origin, t be time, and $(x(t), y(t))$ be the position of the satellite. Then $r = \sqrt{x^2 + y^2}$ is the distance of the object from the center of the earth, $(x'(t), yx'(t))$ is the velocity, $v = \sqrt{(x')^2 + (y')^2}$ is the speed, and $(x''(t), y''(t))$ is the acceleration; see Figure 1. For a more complete discussion of these quantities, see Chapter 9. We now proceed to derive formulas for the two forces acting on the satellite.

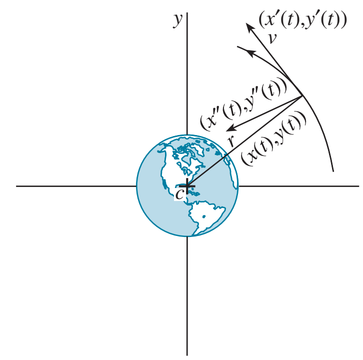

Figure 1 Position, velocity, speed, and acceleration of an object in earth orbit

By Newton's universal law of gravitation, the magnitude of the force due to gravity is $F = \dfrac{m_e m_s g}{r^2}$, where m_e is the mass of the earth, m_s is the mass of the satellite, and g is the gravitational constant. This force is directed toward the center of the earth, and so we find that in terms of components, the vector force due to gravity is $-\dfrac{m_e m_s g}{r^2}\dfrac{(x(t), y(t))}{r}$. This is obtained by multiplying the magnitude by a unit vector from the satellite to the center of the earth.

We will assume that the drag force has magnitude proportional to the product of the surface area A of the object facing the atmosphere, the density of the atmosphere ρ, and the square of velocity v^2. Specifically the magnitude takes the form $CA\rho v^2$, where C is a proportionality constant. One of the most significant assumptions we must make is in the model we take for the atmospheric density.

This density can vary considerably over the surface of the earth, depending upon time of day and year, weather conditions, and other factors. The reasons for variations in density are not particularly well understood, and for our simple model we will assume that atmospheric density depends only upon altitude. Even then, finding a formula for density is not easy. One method is to measure the density at different altitudes experimentally and then find a curve (such as a spline) through the data. This is the approach taken in the example discussed below. Atmospheric scientists have also devised formulas that model the density. One such is

$$\rho = \begin{cases} (1.225 \times 10^{-3})e^{-0.1385(r - r_e)}, & r_e < r < r_e + 90, \\ 10^{1.274 - 4.41[\log_{10}(10.01(r - r_e) - 751.44)]}, & r > r_e + 90 \end{cases}$$

where r_e is the radius of the earth in kilometers.

As we did with the gravitational force, we must write the drag force in terms of components. The action of the force is perpendicular to the forward motion of the object, that is, perpendicular to the velocity. So we obtain the vector force due to drag by multiplying the magnitude of drag by a unit vector tangent to the path, $-CA\rho v^2 \dfrac{(x'(t), y'(t))}{v}$. The negative is due to the drag acting against the direction of the satellite.

Combining these two forces, we get from Newton's second law

$$m_s(x''(t), y''(t)) = -\frac{m_e m_s g}{r^2}\frac{(x(t), y(t))}{r} - CA\rho v^2 \frac{(x'(t), y'(t))}{v}.$$

This reduces to

$$(x''(t), y''(t)) = -\frac{m_e g}{r^3}(x(t), y(t)) - \frac{CA\rho v}{m_s}(x'(t), y'(t)).$$

In terms of components, we get the coupled system of second-order autonomous differential equations

$$x'' = -\frac{m_e g}{r^3}x - \frac{CA\rho v}{m_s}x'$$

$$y'' = -\frac{m_e g}{r^3}y - \frac{CA\rho v}{m_s}y'.$$

To solve this system we transform it to a set of four first-order autonomous differential equations, similar to what was done in Chapter 3. Let $y_1 = x$, $y_2 = x'$, $y_3 = y$, $y_4 = y'$, and $r = \sqrt{y_1^2 + y_3^2}$, $v = \sqrt{y_2^2 + y_4^2}$; then the system to be solved is

$$y_1' = y_2$$

$$y_2' = -\frac{m_e g}{r^3}y_1 - \frac{CA\rho v}{m_s}y_2$$

$$y_3' = y_4$$

$$y_4' = -\frac{m_e g}{r^3}y_3 - \frac{CA\rho v}{m_s}y_4.$$

This system of differential equations is highly nonlinear and cannot be solved analytically. Numerical methods and a computer must be used to get a numerical solution. Figure 2 shows the final one and a half revolutions of a decaying orbit of a satellite closely approximating Skylab. For our solution we used the numerical routines in *Mathematica,* which are similar to, but more sophisticated than, those presented in Chapter 6. The path of the satellite was tracked from a height of 100 kilometers with a velocity of just under 8 km/sec. We see that the satellite follows an elliptical path, spending most of the first revolution at a height greater than 100 kilometers. At the end of the first revolution, the height is approximately 97 km. The rate of decay then increases, and at the end of the next half orbit the height is slightly more than 80 km. Then very quickly the satellite (or what remains of it) strikes the earth. This is reasonable since the density of the atmosphere increases very rapidly as the altitude decreases.

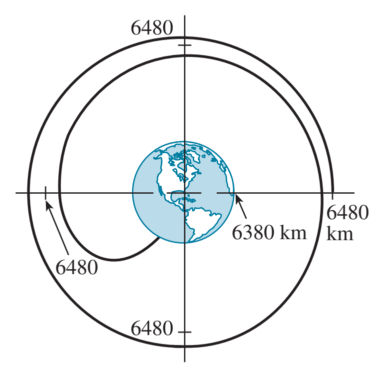

Figure 2 Final orbits of satellite entering earth's atmosphere

Again, it must be stressed that this analysis is based on many assumptions and simplifications. To get a more accurate path, we must improve upon these assumptions—which only makes the solution more difficult to obtain. For these reasons the point of impact of a falling satellite on the earth's surface is hard to predict. We can only hope that it is in the ocean or on uninhabited land.

References

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[3] Heicklen, G., *Atmospheric Chemistry*. New York: Academic Press. 1976.
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